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LARGE ELASTIC DEFORMATIONS OF ISOTROPIC MATERIALS. I. FUNDAMENTAL CONCEPTS

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Introduction

The mathematical theory of small elastic deformations has been developed to a high degree of sophistication on certain fundamental assumptions regarding the stress-strain relationships which are obeyed by the materials considered. The relationships taken are, in effect, a generalization of Hooke's law—*ut tensio, sic vis*. The justification for these assumptions lies in the widespread agreement of experiment with the predictions of the theory and in the interpretation of the elastic behaviour of the materials in terms of their known structure. The same factors have contributed to our appreciation of the limitations of these assumptions.

The principal problems, which the theory seeks to solve, are the determination of the deformation which a body undergoes and the distribution of stresses in it, when certain forces are applied to it, and when certain points of the body are subjected to specified displacements. These problems are always dealt with on the assumption that the generalization of Hooke's law is obeyed by the material of the body and that the deformation is small, i.e. the change of length, in any linear element in the material, is small compared with the length of the element in the undeformed state.

Apart from the fact that the generalization of Hooke's law is obeyed accurately by a very wide range of materials, under a considerable variety of stress and strain conditions, it has the further advantage that it leads to a mathematically tractable theory.

Even if the elastic materials, which it is wished to study, did not obey accurately the simple law postulated but varied in their elastic properties, one from another, there would still be some justification for developing a mathematical theory on this hypothesis. For, it would clearly not be possible to develop a separate mathematical theory, to any considerable extent, based on the known law for each material with which we may be required to deal. The simplicity of the generalization of Hooke's law makes it a peculiarly suitable basis for a mathematical theory which can be regarded as strictly applicable to an ideal material. The departures of the elastic behaviour of any particular material from the generalization of Hooke's law are then regarded as reflexions of the departure of the material from the ideal structure. This would be all the more true if it were found that the generalization of Hooke's law followed from some model of the structure of the elastic materials considered which was satisfactory in other respects.

In seeking a basic hypothesis on which to develop a mathematical theory of large elastic deformations, we are presented with a similar problem. We must choose a law expressing the stress-strain relationships which is sufficiently simple to allow of considerable mathematical development and which at the same time expresses the known behaviour of as wide a range of highly elastic materials as possible. It is quite clear that the wider the range of materials to be embraced by the theory, the less likely will it be that the theory applies exactly to any particular material. It is, however, to be expected that the elastic properties of a group of materials, e.g. the rubber-like materials, having structural similarities, will be expressible by similar laws.

It is necessary, then, to strike a compromise between mathematical tractability, breadth of applicability and exactitude of applicability.

In the present paper, the first steps are taken in the development of a mathematical theory of large elastic deformations.

Part A is principally concerned with the consideration of stress-strain relationships, and their relation to the elastic energy stored in a deformed body. In § 1 the components of strain are defined, in the classical manner (see Love 1927, p. 59 et seq.), when the deformations are no longer small and certain results of these definitions, which are subsequently used in the paper, are recapitulated. In the following section (§ 2) the components of stress are defined. A type of stress-strain relationship, which is shown to be a natural extension of that adopted in the study of small elastic deformations, is studied in some detail (§§ 3 to 6). It is described as neo-Hookean elasticity. The assumption of incompressibility of the material is made in these relationships and in their development. The justification for this is discussed in § 11. This type of stress-strain relationship is shown to have the further merit, as a basis on which to develop a mathematical theory of large elastic deformations, that it is in agreement with that obtained for an ideal rubber-like material, from the molecular statistical considerations of Wall, Flory, Treloar and others. It also shows approximate agreement with measurements on vulcanized rubbers over a wide range of strains, of certain simple types which have been studied experimentally (§ 11).

In developing a mathematical theory by which the deformation of a body resulting from the application of known forces can be calculated, from a knowledge of the elastic behaviour of the material, it is necessary to set up equations of motion and boundary conditions for the body. These may be obtained if either the stress-strain relationships, obeyed by the material of the body, or the elastic energy stored in the body for any specified state of strain, is known. The stress-strain relationships and the formula for the elastically stored energy in terms of strain are related and these relationships are worked out, in general terms, both for a compressible material (§ 7) and for an incompressible material (§ 8). The stored energy formula is then obtained for the special case of neo-Hookean elasticity (§ 9). It is shown that this stored energy formula is equivalent to that normally used for small Hookean elastic deformation, if the assumption of small deformation and incompressibility is introduced (§ 10).

In Part B of the paper, the equations of motion and boundary conditions are obtained both in terms of the stress components (§§ 13, 14) and in terms of the stored energy function (§§ 15, 16). The results obtained by the two methods are formally different but are shown to be equivalent (§§ 17, 18). These results are modified for the case of an incompressible material (§ 19) and the particular case of a neo-Hookean, incompressible material is dealt with in § 20.

Some of the formulae derived in this paper have already been given, particularly by E. & F. Cosserat (1896) and Brillouin (1925). E. & F. Cosserat obtain the equations of equilibrium and boundary conditions for a compressible material, in terms of a general stored energy function. They also quote, from Boussinesq, the relations between the stress components and the stored energy function, derived here in § 7. However, for the sake of completeness and uniformity of treatment and because E. & F. Cosserat's paper is not, in general, readily available for reference, these results are derived afresh here.

E. & F. Cosserat conduct their analysis in a system of rectangular, Cartesian co-ordinates, as is done in this paper, and subsequently generalize the results they obtain to a general system of orthogonal, curvilinear co-ordinates. Brillouin covers the ground already covered by E. & F. Cosserat, but uses the methods of the calculus of tensors and pseudo-tensors to obtain his results. In this paper, it has been decided to conduct the analysis in a system of rectangular, Cartesian co-ordinates and it is intended, in a subsequent paper, to generalize the results obtained to certain particular systems of orthogonal curvilinear co-ordinates which are of special interest in practical problems. Although the procedure of Brillouin has certain advantages of mathematical elegance and generality, it has the disadvantage that the analysis and interpretation of the formulae obtained require a considerable knowledge of the calculus of tensors and pseudo-tensors. Neither E. & F. Cosserat nor Brillouin consider the modifications to their theory which are required in dealing with an incompressible material, nor do they consider the particular law on which the solution of practical problems is to be based.

PART A. STRESS-STRAIN RELATIONSHIPS

1. The definition of large strains

In a fixed, rectangular, Cartesian, co-ordinate system, the co-ordinates of a point of an elastic solid, in its undeformed state, are (x, y, z). In a strained state of the solid, each point (x, y, z) of the solid has undergone displacements whose components parallel to the axes of the co-ordinate system are (u, v, w). (u, v, w) are, in general, functions of (x, y, z). Following Cauchy (1827; Love 1927, pp. 59 et seq.), we may define six components of strain $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{zx}, \epsilon_{xy})$ in the chosen co-ordinate system, thus

$$\left.\begin{aligned} \epsilon_{xx} &= u_x+\tfrac{1}{2}(u_x^2+v_x^2+w_x^2),\\ \epsilon_{yy} &= v_y+\tfrac{1}{2}(u_y^2+v_y^2+w_y^2),\\ \epsilon_{zz} &= w_z+\tfrac{1}{2}(u_z^2+v_z^2+w_z^2),\\ \epsilon_{yz} &= w_y+v_z+u_y u_z+v_y v_z+w_y w_z,\\ \epsilon_{zx} &= u_z+w_x+u_z u_x+v_z v_x+w_z w_x,\\ \epsilon_{xy} &= v_x+u_y+u_x u_y+v_x v_y+w_x w_y. \end{aligned}\right\} \tag{1·1}$$

It is clear that when the deformations are small, so that second degree terms in $u_x, u_y, \ldots, w_z$ can be neglected in comparison with first degree terms, the components of strain $(\epsilon_{xx}, \epsilon_{yy}, \ldots, \epsilon_{xy})$ become $(e_{xx}, e_{yy}, \ldots, e_{xy})$, the components of strain for small deformations, where

$$\left.\begin{aligned} e_{xx} &= u_x, & e_{yy} &= v_y, & e_{zz} &= w_z,\\ e_{yz} &= v_z+w_y, & e_{zx} &= w_x+u_z, & e_{xy} &= u_y+v_x. \end{aligned}\right\} \tag{1·2}$$

The length ds', in the deformed state, of a linear element of the material, which in the undeformed state has a length ds, lies at the point (x, y, z) and has direction-cosines (l, m, n) with respect to the co-ordinate system, is given by

$$\left(\frac{ds'}{ds}\right)^2 = (1+2\epsilon_{xx})\,l^2+(1+2\epsilon_{yy})\,m^2+(1+2\epsilon_{zz})\,n^2+2\epsilon_{yz}\,mn+2\epsilon_{zx}\,nl+2\epsilon_{xy}\,lm. \tag{1·3}$$

Thus, if the six components of the strain, as defined by Cauchy, are given at a point, the ratio of the deformed to undeformed length of an element at that point, having any specified direction in the unstrained state, may be calculated. In particular $(1+2\epsilon_{xx})$, $(1+2\epsilon_{yy})$ and $(1+2\epsilon_{zz})$ give respectively the values of $(ds'/ds)^2$ for an element of length which lies at (x,y,z) and has a direction parallel to the x, y or z axes in the undeformed state.

From (1·3) the reciprocal strain ellipsoid

$$(1+2\epsilon_{xx})\,x^2+(1+2\epsilon_{yy})\,y^2+(1+2\epsilon_{zz})\,z^2+2\epsilon_{yz}yz+2\epsilon_{zx}zx+2\epsilon_{xy}xy=1$$

may be defined at any point of the material. It has the property that the length of its radius in any direction is proportional to ds/ds', for an element which lies in that direction in the undeformed state.

In a similar manner, the strain ellipsoid may be defined. This has the property that the length of its radius in any direction is proportional to ds'/ds, for an element which lies in that direction in the deformed state.

Elements of length, which are parallel to the axes of the reciprocal strain ellipsoid in the undeformed state, become parallel to the axes of the strain ellipsoid in the deformed state. Such elements, all having lengths ds in the undeformed state, have lengths $\lambda_1 ds$, $\lambda_2 ds$ and $\lambda_3 ds$ respectively in the deformed state, where λ_1-1, λ_2-1 and λ_3-1 are called the principal extensions and λ_1^2, λ_2^2, λ_3^2 are given by the roots of the equation

$$\begin{vmatrix} 1+2\epsilon_{xx}-\lambda^2 & \epsilon_{xy} & \epsilon_{zx} \\ \epsilon_{xy} & 1+2\epsilon_{yy}-\lambda^2 & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{yz} & 1+2\epsilon_{zz}-\lambda^2 \end{vmatrix}=0. \tag{1·4}$$

The direction-cosines (l',m',n') of an element of length ds' in the deformed state can be defined, in terms of its direction-cosines (l,m,n) and length ds in the undeformed state, by the relations

$$\left.\begin{aligned} l' &= \frac{ds}{ds'}[(1+u_x)\,l+u_y m+u_z n], \\ m' &= \frac{ds}{ds'}[v_x l+(1+v_y)\,m+v_z n], \\ n' &= \frac{ds}{ds'}[w_x l+w_y m+(1+w_z)\,n]. \end{aligned}\right\} \tag{1·5}$$

Also, an element of volume dV situated at (x,y,z) in the undeformed state has a volume $\tau\,dV$ in the deformed state, where

$$\tau=\begin{vmatrix} 1+u_x & u_y & u_z \\ v_x & 1+v_y & v_z \\ w_x & w_y & 1+w_z \end{vmatrix}. \tag{1·6}$$

For an incompressible material, the volumes of an element in the deformed and undeformed states are equal, i.e.

$$\tau=1. \tag{1·7}$$

2. The specification of stress

In this paper, the Saint-Venant notation for stress will be used. Thus, the stress component t_{xx} denotes the force parallel to the x-axis, per unit area of the deformed material, which in the deformed state is normal to the x-axis. The stress components t_{yy} and t_{zz} are similarly defined. The stress component t_{yz} denotes the force parallel to the y-axis, per unit area of the deformed material, which in the deformed state is normal to the z-axis. The stress components t_{zy}, t_{zx}, t_{xz}, t_{xy} and t_{yx} are similarly defined. All the stress components refer to stresses at a point of the material which is at (x, y, z) in the undeformed state.

Although we have defined nine stress components, only six of these are independent, on account of the three relations (Love 1927, § 47) of the type $t_{yz} = t_{zy}$.

3. The stress-strain relationships for an incompressible material

The generalization of Hooke's law, used in the mathematical theory of small elastic deformations of isotropic materials, may be expressed by the six relations

$$\left.\begin{aligned} e_{xx} &= (1/E)\,[(1+\sigma)\,t_{xx}-\sigma(t_{xx}+t_{yy}+t_{zz})], \quad \text{etc.} \\ \text{and} \qquad e_{yz} &= (2/E)\,(1+\sigma)\,t_{yz}, \quad \text{etc.,} \end{aligned}\right\} \tag{3·1}$$

where E is Young's modulus and σ is Poisson's ratio. $(e_{xx}, e_{yy}, \ldots, e_{xy})$ are the components of strain for the small deformation.

The expression $(\sigma/1+\sigma)\,(t_{xx}+t_{yy}+t_{zz})$ in the first three of equations (3·1) is equivalent to a hydrostatic pressure p (say). Also, for an incompressible material, we have approximately

$$e_{xx}+e_{yy}+e_{zz} = 0.$$

Therefore $$\sigma = \tfrac{1}{2}.$$

Equations (3·1) then become

$$e_{xx} = (3/2E)\,(t_{xx}-p), \text{ etc.,} \quad \text{and} \quad e_{yz} = (3/E)\,t_{yz}, \text{ etc.} \tag{3·2}$$

If the strains are specified, then the pressure p in equations (3·2) may have an arbitrary value, but if the stresses are specified, p is determined by the relation

$$p = \tfrac{1}{3}(t_{xx}+t_{yy}+t_{zz}).$$

Equations (3·2) constitute a generalized statement of Hooke's law for small deformations of an incompressible isotropic material.

In choosing the stress-strain relationships for a highly elastic and incompressible isotropic material, there are certain limitations which must be observed. We note that any homogeneous strain—and the most general strain may be considered homogeneous over an infinitesimally small region of space—may be considered to consist of a rotation, in which the axes of the reciprocal strain ellipsoid are brought into the position of the axes of the strain ellipsoid, followed by a pure homogeneous strain.

Now, in the rotation, no work is done on the element considered, and consequently stress-strain relationships for the most general possible deformation may be derived from a knowledge of the stress-strain relationships for a pure, homogeneous deformation. The actual form of this relationship will depend on the structure of the material considered, but for mathematical simplicity let us consider the case where stress is related to strain, in a pure, homogeneous deformation, by the law

$$\left.\begin{aligned}1+2\epsilon_{XX} &= (3/E)(t_{XX}-p), \quad 1+2\epsilon_{YY} = (3/E)(t_{YY}-p),\\ 1+2\epsilon_{ZZ} &= (3/E)(t_{ZZ}-p), \quad \epsilon_{YZ}=\epsilon_{ZX}=\epsilon_{XY}=0,\end{aligned}\right\} \tag{3·3}$$

where X, Y and Z are the axes of the strain ellipsoid.

If, in the stress-strain relationships (3·2) for small elastic deformations, the axes X, Y and Z of the strain ellipsoid are chosen as co-ordinate axes, we have

$$\left.\begin{aligned}2e_{XX} &= (3/E)(t_{XX}-p), \qquad 2e_{YY} = (3/E)(t_{YY}-p),\\ 2e_{ZZ} &= (3/E)(t_{ZZ}-p), \quad \text{and} \quad e_{YZ}=e_{ZX}=e_{XY}=0.\end{aligned}\right\} \tag{3·4}$$

It is seen that equations (3·4) are similar to equations (3·3) if $e_{XX}, e_{YY}, \ldots$, the strain components for small deformations, are replaced by $\epsilon_{XX}, \epsilon_{YY}, \ldots$, the strain components for large deformations defined in §1. Thus, equations (3·3) form a *natural generalization of Hooke's law to the case of large strains*, and a material obeying such a laws will be described as an incompressible, neo-Hookean material.

It will be seen that although p, in equations (3·3), has the nature of a negative hydrostatic pressure, it is no longer equal to $\frac{1}{3}(t_{XX}+t_{YY}+t_{ZZ})$—the mean tension at the point—as in the theory of small elastic deformations. It can, however, be determined in terms of the stress components, as explained in §5. We shall, for convenience, refer to p as the hydrostatic pressure at the point, although it should be remarked that any stress can be resolved into a hydrostatic pressure and another stress in an infinite number of ways. The expression $\frac{1}{3}(t_{XX}+t_{YY}+t_{ZZ})$ used in the classical theory for small elastic deformations is simply one choice of this hydrostatic pressure, which loses its mathematical convenience when large strains are considered. In the present theory it becomes more convenient to choose the hydrostatic pressure as has been done here. It will be noted, however, that when the body is undeformed and all the stress components are zero, $p=-\frac{1}{3}E$. This choice of the hydrostatic pressure has no particular physical significance, but is merely mathematically convenient.

For a pure, homogeneous deformation, suppose that λ_1, λ_2, λ_3 are the lengths in the deformed state of linear elements parallel to the axes X, Y, Z respectively, which have unit length in the undeformed state.

Then, by making (l, m, n) successively $(1, 0, 0)$ $(0, 1, 0)$ and $(0, 0, 1)$ in equation (1·3) we see that

$$\lambda_1^2 = 1+2\epsilon_{XX}, \quad \lambda_2^2 = 1+2\epsilon_{YY} \quad \text{and} \quad \lambda_3^2 = 1+2\epsilon_{ZZ}.$$

Equations (3·3) may therefore be rewritten

$$\left.\begin{aligned}t_{XX} &= \tfrac{1}{3}E\lambda_1^2+p, \qquad t_{YY} = \tfrac{1}{3}E\lambda_2^2+p,\\ t_{ZZ} &= \tfrac{1}{3}E\lambda_3^2+p \quad \text{and} \quad t_{YZ}=t_{ZX}=t_{XY}=0.\end{aligned}\right\} \tag{3·5}$$

It has been pointed out already that a general deformation in an infinitesimal region may be considered to consist of a rotation together with a pure, homogeneous strain. The principal extensions λ_1-1, λ_2-1, λ_3-1, in this pure, homogeneous strain, are the same as those for the general, inhomogeneous strain, of which it forms a part. The stress-strain relations (3·5) will thus apply to the general deformation of an isotropic body. They may be transformed to refer to any fixed, rectangular, Cartesian system of co-ordinates (x, y, z).

Suppose the direction-cosines of the axes (X, Y, Z) of the strain ellipsoid, referred to the co-ordinate system (x, y, z), are given by

	X	Y	Z
x	l'_1	l'_2	l'_3
y	m'_1	m'_2	m'_3
z	n'_1	n'_2	n'_3

Then the components of the stress, referred to the axes (x, y, z), are given, in terms of those referred to the axes (X, Y, Z), by the equations

$$\left.\begin{aligned} t_{xx} &= l_1'^2 t_{XX} + l_2'^2 t_{YY} + l_3'^2 t_{ZZ}, \\ t_{yy} &= m_1'^2 t_{XX} + m_2'^2 t_{YY} + m_3'^2 t_{ZZ}, \\ t_{zz} &= n_1'^2 t_{XX} + n_2'^2 t_{YY} + n_3'^2 t_{ZZ}, \\ t_{yz} &= m_1' n_1' t_{XX} + m_2' n_2' t_{YY} + m_3' n_3' t_{ZZ}, \\ t_{zx} &= n_1' l_1' t_{XX} + n_2' l_2' t_{YY} + n_3' l_3' t_{ZZ}, \\ t_{xy} &= l_1' m_1' t_{XX} + l_2' m_2' t_{YY} + l_3' m_3' t_{ZZ}. \end{aligned}\right\} \tag{3·6}$$

If (l_i, m_i, n_i), for $i = 1, 2, 3$, are the direction-cosines, referred to the co-ordinate axes (x, y, z), of the axes of the reciprocal strain ellipsoid, corresponding to the axes of the strain ellipsoid whose direction-cosines are (l'_i, m'_i, n'_i), then

$$\left.\begin{aligned} l'_i &= (1/\lambda_i)\,[(1+u_x)\,l_i + u_y m_i + u_z n_i], \\ m'_i &= (1/\lambda_i)\,[v_x l_i + (1+v_y)\,m_i + v_z n_i] \\ \text{and} \qquad n'_i &= (1/\lambda_i)\,[w_x l_i + w_y m_i + (1+w_z)\,n_i]. \end{aligned}\right\} \tag{3·7}$$

Substituting in equations (3·6) the expressions for (l'_i, m'_i, n'_i), given in equations (3·7), and the expressions for (t_{XX}, t_{YY}, t_{ZZ}), given by equations (3·5), and bearing in mind that

$$l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1$$

and $$l_1 m_1 + l_2 m_2 + l_3 m_3 = m_1 n_1 + m_2 n_2 + m_3 n_3 = n_1 l_1 + n_2 l_2 + n_3 l_3 = 0,$$

the following 'stress-strain' relationships are obtained:

$$\left.\begin{aligned} t_{xx} &= \tfrac{1}{3}E[(1+u_x)^2 + u_y^2 + u_z^2] + p, \\ t_{yy} &= \tfrac{1}{3}E[v_x^2 + (1+v_y)^2 + v_z^2] + p, \\ t_{zz} &= \tfrac{1}{3}E[w_x^2 + w_y^2 + (1+w_z)^2] + p, \\ t_{yz} &= \tfrac{1}{3}E[v_x w_x + (1+v_y)\,w_y + v_z(1+w_z)], \\ t_{zx} &= \tfrac{1}{3}E[w_x(1+u_x) + w_y u_y + (1+w_z)\,u_z], \\ t_{xy} &= \tfrac{1}{3}E[(1+u_x)\,v_x + u_y(1+v_y) + u_z v_z]. \end{aligned}\right\} \tag{3·8}$$

We see that the components of stress are, in general, no longer functions of the components of strain, as defined in § 1, but are given by other functions of the nine quantities $(u_x, u_y, u_z, \ldots)$. It can readily be shown that

$$\left.\begin{aligned} (1+u_x)^2+u_y^2+u_z^2 &= 1+2\epsilon'_{xx},\\ v_x^2+(1+v_y)^2+v_z^2 &= 1+2\epsilon'_{yy},\\ w_x^2+w_y^2+(1+w_z)^2 &= 1+2\epsilon'_{zz},\\ v_x w_x+(1+v_y)\,w_y+v_z(1+w_z) &= \epsilon'_{yz},\\ w_x(1+u_x)+w_y u_y+(1+w_z)\,u_z &= \epsilon'_{zx},\\ (1+u_x)\,v_x+u_y(1+v_y)+u_z v_z &= \epsilon'_{xy}, \end{aligned}\right\} \tag{3·9}$$

where $\epsilon'_{xx}, \epsilon'_{yy}, \epsilon'_{zz}, \ldots$ are the components, relative to the co-ordinate system (x, y, z), of the pure, homogeneous portion of the strain.

It may be noted that for a pure, homogeneous strain, we have (Love 1927, § 33) the relationships

$$v_z = w_y, \quad w_x = u_z \quad \text{and} \quad u_y = v_x.$$

Introducing these into the expressions for $(\epsilon'_{xx}, \epsilon'_{yy}, \ldots, \epsilon'_{yz})$ we obtain $(\epsilon_{xx}, \epsilon_{yy}, \ldots, \epsilon_{yz})$ respectively.

Introducing (3·9) into equations (3·8), we have

$$\left.\begin{aligned} t_{xx} &= \tfrac{1}{3}E(1+2\epsilon'_{xx})+p, & t_{yz} &= \tfrac{1}{3}E\epsilon'_{yz},\\ t_{yy} &= \tfrac{1}{3}E(1+2\epsilon'_{yy})+p, & t_{zx} &= \tfrac{1}{3}E\epsilon'_{zx},\\ t_{zz} &= \tfrac{1}{3}E(1+2\epsilon'_{zz})+p, & t_{xy} &= \tfrac{1}{3}E\epsilon'_{xy}. \end{aligned}\right\} \tag{3·10}$$

From these equations it is seen that, if the components of the pure, homogeneous portion of the strain are specified, the normal stresses t_{xx}, t_{yy} and t_{zz} are undetermined to the extent of an arbitrary hydrostatic pressure p. On the other hand, if the components of stress are specified, the hydrostatic pressure is no longer arbitrary and can be determined. Hence the components of the pure, homogeneous portion of the strain can be found, in the manner described in § 5.

4. The stress-strain relationships in simple extension and simple shear

(a) *Simple extension*

For the simple extension parallel to the x-axis of a body, the material of which is neo-Hookean and incompressible, the stress-strain relationships (3·5) take the form

$$t_{xx} = \tfrac{1}{3}E\lambda_1^2+p, \quad t_{yy} = \tfrac{1}{3}E\lambda_2^2+p = 0,$$
$$t_{zz} = \tfrac{1}{3}E\lambda_3^2+p = 0 \quad \text{and} \quad t_{yz} = t_{zx} = t_{xy} = 0.$$

Whence, since $\lambda_1\lambda_2\lambda_3 = 1$, we have

$$\lambda_2^2 = \lambda_3^2 = \frac{1}{\lambda_1} \quad \text{and} \quad p = -\tfrac{1}{3}E\frac{1}{\lambda_1}.$$

This gives

$$t_{xx} = \tfrac{1}{3}E\left(\lambda_1^2 - \frac{1}{\lambda_1}\right).$$

(*b*) *Simple shear*

For a simple shear, in which laminae of the material in the xy-plane move parallel to the x-axis, u is a function of z only and $v = w = 0$; i.e.

$$u_z \neq 0 \quad \text{and} \quad u_x = u_y = v_x = v_y = v_z = w_x = w_y = w_z = 0.$$

Introducing these results into equations (3·8), we have the stress-strain relationships for the simple shear

$$t_{xx} = \tfrac{1}{3}E(1+u_z^2)+p, \quad t_{yy} = t_{zz} = \tfrac{1}{3}E+p,$$
$$t_{yz} = 0, \quad t_{zx} = \tfrac{1}{3}Eu_z \quad \text{and} \quad t_{xy} = 0.$$

From these equations it can be seen that *shearing stresses alone cannot maintain a state of simple shear in the material*. If the stresses t_{yy} and t_{zz} are zero, then the stress $t_{xx} = \frac{1}{3}Eu_z^2$ and if the stress $t_{xx} = 0$, $t_{yy} = t_{zz} = -\frac{1}{3}Eu_z^2$. Thus, two of the possible stress systems which can maintain a simple shear u_z in the material are:

(i) a shearing stress $\frac{1}{3}Eu_z$ in the xz-plane, together with a normal stress $\frac{1}{3}Eu_z^2$ parallel to the x-axis;

(ii) a shearing stress $\frac{1}{3}Eu_z$ in the xz-plane, together with two normal stresses each of amount $-\frac{1}{3}Eu_z^2$ parallel to the y- and z-axes.

If the amount of shear is small (i.e. u_z^2 can be neglected), we obtain agreement with the theory for small deformations of a Hookean material, in which a simple shear may be maintained by a shearing force alone.

5. The determination of the hydrostatic pressure for given stress components

From (1·6) and (1·7), it follows that

$$\tau^2 = \begin{vmatrix} 1+u_x & u_y & u_z \\ v_x & 1+v_y & v_z \\ w_x & w_y & 1+w_z \end{vmatrix} \times \begin{vmatrix} 1+u_x & v_x & w_x \\ u_y & 1+v_y & w_y \\ u_z & v_z & 1+w_z \end{vmatrix}$$

and $\tau^2 = 1$, for an incompressible material.

Carrying out the multiplication of the determinants and making use of the relations (3·9), it follows that

$$\begin{vmatrix} 1+2\epsilon'_{xx} & \epsilon'_{xy} & \epsilon'_{zx} \\ \epsilon'_{xy} & 1+2\epsilon'_{yy} & \epsilon'_{yz} \\ \epsilon'_{zx} & \epsilon'_{yz} & 1+2\epsilon'_{zz} \end{vmatrix} = 1. \tag{5·1}$$

Substituting in (5·1) for $(1+2\epsilon'_{xx}), \ldots, \epsilon'_{yz}, \ldots$, from equations (3·10), we have

$$\begin{vmatrix} t_{xx}-p & t_{xy} & t_{zx} \\ t_{xy} & t_{yy}-p & t_{yz} \\ t_{zx} & t_{yz} & t_{zz}-p \end{vmatrix} = \tfrac{1}{27}E^3. \tag{5·2}$$

This enables us to determine the hydrostatic pressure p corresponding to a specified system of stress components and thus, from equations (3·10), to find the components of the pure, homogeneous portion of the strain.

Equation (5·2) may be rewritten

$$p^3-(t_{xx}+t_{yy}+t_{zz})\,p^2+(t_{yy}t_{zz}+t_{zz}t_{xx}+t_{xx}t_{yy}-t_{yz}^2-t_{zx}-t_{xy}^2)\,p$$
$$-(t_{xx}t_{yy}t_{zz}+2t_{yz}t_{zx}t_{xy}-t_{xx}t_{yz}^2-t_{yy}t_{zx}^2-t_{zz}t_{xy}^2) = -\tfrac{1}{27}E^3. \tag{5·3}$$

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This is a cubic equation in p and consequently has either one or three real roots, depending on the values of the quantities

$$(t_{xx}+t_{yy}+t_{zz}), \quad (t_{yy}t_{zz}+t_{zz}t_{xx}+t_{xx}t_{yy}-t_{yz}^2-t_{zx}^2-t_{xy}^2)$$

and
$$(t_{xx}t_{yy}t_{zz}+2t_{yz}t_{zx}t_{xy}-t_{xx}t_{yz}^2-t_{yy}t_{zx}^2-t_{zz}t_{xy}^2-\tfrac{1}{27}E^3).$$

It might at first sight appear that in the case when equation (5·3) has three real roots, giving three real values of p, there are, from equations (3·10), three possible sets of values of the strain components corresponding to a given set of values of the stress components. It can, however, be readily shown that this is not the case.

6. Uniqueness of the strain components for given stress components

The coefficients of $-p^2$, p and $-p^0$ on the left-hand side of equation (5·3) are invariants of the stress components as regards orthogonal transformations of co-ordinates.

If the axes of the strain ellipsoid (X, Y, Z) are taken as co-ordinate axes, we have $t_{YZ} = t_{ZX} = t_{XY} = 0$, and equation (5·3) becomes

$$p^3-(t_{XX}+t_{YY}+t_{ZZ})\,p^2+(t_{YY}t_{ZZ}+t_{ZZ}t_{XX}+t_{XX}t_{YY})\,p-t_{XX}t_{YY}t_{ZZ} = -\tfrac{1}{27}E^3,$$

or
$$(p-t_{XX})\,(p-t_{YY})\,(p-t_{ZZ})+\tfrac{1}{27}E^3 = 0. \qquad (6·1)$$

Equation (6·1) can, of course, be obtained directly from equations (3·5), noting that $\lambda_1\lambda_2\lambda_3 = 1$ for an incompressible material.

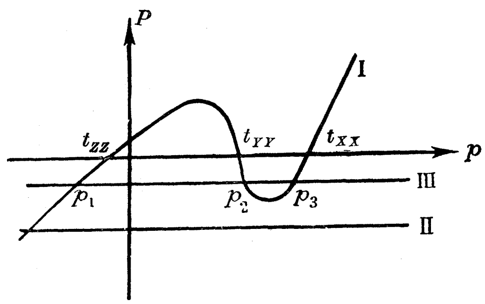

Figure 1

The distribution of the values of p satisfying equation (6·1) with respect to the values of t_{XX}, t_{YY} and t_{ZZ} can be determined in the following way. In order to fix our ideas let us assume that $t_{XX} > t_{YY} > t_{ZZ}$.

The solutions of equation (6·1) occur at the values of p for which the curve

$$P = (p-t_{XX})\,(p-t_{YY})\,(p-t_{ZZ}) \qquad (6·2)$$

intersects the straight line parallel to the p-axis

$$P = -\tfrac{1}{27}E^3. \qquad (6·3)$$

We note that the curve (6·2) intersects the p-axis when $p = t_{XX}$, t_{YY} and t_{ZZ} and that when $p \to \infty$, $P \to \infty$ and when $p \to -\infty$, $P \to -\infty$. Also, we note that P is a single-valued function of p. Consequently the curve (6·2) has the general form of curve I in figure 1.

Noting that E is always positive and that therefore the straight line (6·3) always lies below the p-axis in figure 1, we see that, for sufficiently large values of E, the straight line

(6·3) will occupy such a position as II, giving one point of intersection with curve I. Therefore, for sufficiently large values of E, the equation (6·1) has only one real solution. For smaller values of E, the straight line (6·1) may occupy such a position as III, and then equation (6·1) has three real solutions, given by the values p_1, p_2 and p_3 of p for which the curve I intersects the straight line III.

It is quite clear, however, that one of these values p_1 must be less than t_{ZZ}, i.e. less than any of the three stress components t_{XX}, t_{YY}, t_{ZZ}, while the other two values p_2 and p_3 must be between t_{YY} and t_{XX}.

Equations (3·5) may be rewritten

$$\tfrac{1}{3}E\lambda_1^2 = t_{XX}-p, \quad \tfrac{1}{3}E\lambda_2^2 = t_{YY}-p,$$
$$\tfrac{1}{3}E\lambda_3^2 = t_{ZZ}-p, \quad t_{YZ} = t_{ZX} = t_{XY} = 0.$$

Consequently, if $p = p_1$, λ_1^2, λ_2^2 and λ_3^2 are all positive, giving rise to real values for λ_1, λ_2 and λ_3. Whereas, if $p = p_2$ or p_3, λ_2^2 and λ_3^2 are both negative, giving rise to imaginary values for λ_2 and λ_3. Thus the values p_2 and p_3 for p do not correspond to real states of deformation, and, for given stress components, the pure, homogeneous portion of the deformation is uniquely determined.

7. The stored energy function

When an elastic body is strained, energy is stored in it. For small deformations of the body, within the scope of the classical mathematical theory, this energy is determined completely by the strain components $(e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy})$. When the deformations are large, the energy stored in an element of the body is determined by the six components $(\epsilon'_{xx}, \epsilon'_{yy}, \epsilon'_{zz}, \ldots)$ of the pure, homogeneous strain, which, together with a pure rotation, constitutes the strain undergone by the element of the body. In the purely rotational part of the strain no energy is stored in the element considered.

Since $\epsilon'_{xx}, \epsilon'_{yy}, \epsilon'_{zz}, \ldots$ are functions of the nine quantities $u_x, u_y, u_z, \ldots, w_z$ specified by equations (3·9), the stored energy may be considered to be a function of these nine quantities.

In order to find an expression for the stored energy in terms of these nine quantities, if the stress-strain relationships for the material are specified, consider an elementary cuboid of the strained material, situated at $(\xi, \eta, \zeta) = (x+u, y+v, z+w)$, whose edges are parallel to the co-ordinate axes (x, y, z) and have lengths $(\Delta\xi, \Delta\eta, \Delta\zeta)$.

If the displacement configuration (u, v, w) undergoes a small change $(\delta u, \delta v, \delta w)$, then the edges of the cuboid undergo changes of length $\delta(\Delta u)$, $\delta(\Delta v)$, $\delta(\Delta w)$, where

$$\Delta\xi = \Delta x + \Delta u, \quad \Delta\eta = \Delta y + \Delta v \quad \text{and} \quad \Delta\zeta = \Delta z + \Delta w.$$

Second and higher degree terms in $\delta(\Delta u)$, $\delta(\Delta v)$ and $\delta(\Delta w)$ are neglected.

The forces acting over these displacements are

$$t_{xx}\Delta\eta\,\Delta\zeta + t_{xy}\Delta\zeta\,\Delta\xi + t_{xz}\Delta\xi\,\Delta\eta,$$
$$t_{yx}\Delta\eta\,\Delta\zeta + t_{yy}\Delta\zeta\,\Delta\xi + t_{yz}\Delta\xi\,\Delta\eta$$

and

$$t_{zx}\Delta\eta\,\Delta\zeta + t_{zy}\Delta\zeta\,\Delta\xi + t_{zz}\Delta\xi\,\Delta\eta,$$

respectively. The total work done in the virtual, relative displacements $(\delta(\Delta u), \delta(\Delta v), \delta(\Delta w))$ is thus

$$\begin{aligned}\delta W_1 = &(t_{xx}\Delta\eta\,\Delta\zeta + t_{xy}\Delta\zeta\,\Delta\xi + t_{xz}\Delta\xi\,\Delta\eta)\,\delta(\Delta u)\\ &+(t_{yx}\Delta\eta\,\Delta\zeta + t_{yy}\Delta\zeta\,\Delta\xi + t_{yz}\Delta\xi\,\Delta\eta)\,\delta(\Delta v)\\ &+(t_{zx}\Delta\eta\,\Delta\zeta + t_{zy}\Delta\zeta\,\Delta\xi + t_{zz}\Delta\xi\,\Delta\eta)\,\delta(\Delta w).\end{aligned} \tag{7·1}$$

Equation (7·1) may be rewritten

$$\begin{aligned}\delta W_1 = \tau\Big[&t_{xx}\frac{\delta(\Delta u)}{\Delta\xi} + t_{xy}\frac{\delta(\Delta u)}{\Delta\eta} + t_{xz}\frac{\delta(\Delta u)}{\Delta\zeta}\\ &+t_{yx}\frac{\delta(\Delta v)}{\Delta\xi} + t_{yy}\frac{\delta(\Delta v)}{\Delta\eta} + t_{yz}\frac{\delta(\Delta v)}{\Delta\zeta}\\ &+t_{zx}\frac{\delta(\Delta w)}{\Delta\xi} + t_{zy}\frac{\delta(\Delta w)}{\Delta\eta} + t_{zz}\frac{\delta(\Delta w)}{\Delta\zeta}\Big]\Delta x\,\Delta y\,\Delta z.\end{aligned} \tag{7·2}$$

If W is the elastically stored energy in the deformed state, per unit volume (measured in the undeformed state),

$$\begin{aligned}\delta W = &\frac{\partial W}{\partial u_\xi}\frac{\delta(\Delta u)}{\Delta\xi} + \frac{\partial W}{\partial u_\eta}\frac{\delta(\Delta u)}{\Delta\eta} + \frac{\partial W}{\partial u_\zeta}\frac{\delta(\Delta u)}{\Delta\zeta}\\ &+\frac{\partial W}{\partial v_\xi}\frac{\delta(\Delta v)}{\Delta\xi} + \frac{\partial W}{\partial v_\eta}\frac{\delta(\Delta v)}{\Delta\eta} + \frac{\partial W}{\partial v_\zeta}\frac{\delta(\Delta v)}{\Delta\zeta}\\ &+\frac{\partial W}{\partial w_\xi}\frac{\delta(\Delta w)}{\Delta\xi} + \frac{\partial W}{\partial w_\eta}\frac{\delta(\Delta w)}{\Delta\eta} + \frac{\partial W}{\partial w_\zeta}\frac{\delta(\Delta w)}{\Delta\zeta},\end{aligned} \tag{7·3}$$

where $\partial W/\partial u_\xi$ is defined as $\lim\limits_{\substack{\delta u\to 0\\ \Delta\xi\to 0}} \delta W\Big/\dfrac{\Delta(\delta u)}{\Delta\xi}$, with similar definitions for $\partial W/\partial u_\eta$, etc. It is noteworthy that $\delta W\Big/\dfrac{\Delta(\delta u)}{\Delta\xi}$ is not equal to $\delta W\Big/\delta\left(\dfrac{\Delta u}{\Delta\xi}\right)$, since $\Delta\xi$ is dependent on Δu. Thus, $\lim\limits_{\substack{\delta u\to 0\\ \Delta\xi\to 0}} \delta W\Big/\dfrac{\Delta(\delta u)}{\Delta\xi}$ is not equal to $\partial W\Big/\partial\left(\dfrac{\partial u}{\partial\xi}\right)$.

The energy stored elastically in the element of volume considered, as a result of the virtual displacement, is

$$\delta W\,\Delta x\,\Delta y\,\Delta z.$$

For equilibrium,

$$\delta W\,\Delta x\,\Delta y\,\Delta z = \delta W_1, \tag{7·4}$$

for all allowable configurations of the quantities $(\delta(\Delta u), \delta(\Delta v), \delta(\Delta w))$ and for all ratios between $\Delta\xi$, $\Delta\eta$ and $\Delta\zeta$, where δW_1 and δW are given by equations (7·2) and (7·3) respectively.

Consequently,

$$\begin{aligned}t_{xx} &= \frac{1}{\tau}\frac{\partial W}{\partial u_\xi}, & t_{xy} &= \frac{1}{\tau}\frac{\partial W}{\partial u_\eta}, & t_{xz} &= \frac{1}{\tau}\frac{\partial W}{\partial u_\zeta},\\ t_{yx} &= \frac{1}{\tau}\frac{\partial W}{\partial v_\xi}, & t_{yy} &= \frac{1}{\tau}\frac{\partial W}{\partial v_\eta}, & t_{yz} &= \frac{1}{\tau}\frac{\partial W}{\partial v_\zeta},\\ t_{zx} &= \frac{1}{\tau}\frac{\partial W}{\partial w_\xi}, & t_{zy} &= \frac{1}{\tau}\frac{\partial W}{\partial w_\eta}, & t_{zz} &= \frac{1}{\tau}\frac{\partial W}{\partial w_\zeta}.\end{aligned}$$

Let us consider, for example, the first of these relations. From the definition of $\partial W/\partial u_\xi$,

$$t_{xx} = \frac{1}{\tau} \lim_{\substack{\delta u \to 0 \\ \Delta\xi \to 0}} \delta W \Big/ \frac{\Delta(\delta u)}{\Delta \xi}.$$

Now, since
$$\xi = x + u,$$

$$\Delta\xi = (1+u_x)\,\Delta x + u_y\,\Delta y + u_z\,\Delta z.$$

Thus,
$$t_{xx} = \frac{1}{\tau} \lim_{\substack{\delta u \to 0 \\ \Delta x,\, \Delta y,\, \Delta z \to 0}} \frac{\delta W}{\Delta(\delta u)} [(1+u_x)\,\Delta x + u_y\,\Delta y + u_z\,\Delta z]$$
$$= \frac{1}{\tau}\left[(1+u_x)\frac{\partial W}{\partial u_x} + u_y \frac{\partial W}{\partial u_y} + u_z \frac{\partial W}{\partial u_z}\right].$$

Here, $\partial W/\partial u_x$ denotes $\partial W \Big/ \partial\left(\dfrac{\partial u}{\partial x}\right)$, $\partial W/\partial u_y$ denotes $\partial W \Big/ \partial\left(\dfrac{\partial u}{\partial y}\right)$, and so on.

It should be borne in mind that

$$\lim_{\substack{\delta u \to 0 \\ \Delta x \to 0}} \delta W \Big/ \frac{\Delta(\delta u)}{\Delta x} = \lim_{\substack{\delta u \to 0 \\ \Delta x \to 0}} \delta W \Big/ \delta\left(\frac{\Delta u}{\Delta x}\right),$$

since Δx is independent of δu.

In a similar manner, expressions for the other components of stress can be found. Thus

$$\left.\begin{aligned}
t_{xx} &= \frac{1}{\tau}\left[(1+u_x)\frac{\partial W}{\partial u_x} + u_y \frac{\partial W}{\partial u_y} + u_z \frac{\partial W}{\partial u_z}\right],\\
t_{xy} &= \frac{1}{\tau}\left[v_x \frac{\partial W}{\partial u_x} + (1+v_y)\frac{\partial W}{\partial u_y} + v_z \frac{\partial W}{\partial u_z}\right],\\
t_{xz} &= \frac{1}{\tau}\left[w_x \frac{\partial W}{\partial u_x} + w_y \frac{\partial W}{\partial u_y} + (1+w_z)\frac{\partial W}{\partial u_z}\right],\\
t_{yx} &= \frac{1}{\tau}\left[(1+u_x)\frac{\partial W}{\partial v_x} + u_y \frac{\partial W}{\partial v_y} + u_z \frac{\partial W}{\partial v_z}\right],\\
t_{yy} &= \frac{1}{\tau}\left[v_x \frac{\partial W}{\partial v_x} + (1+v_y)\frac{\partial W}{\partial v_y} + v_z \frac{\partial W}{\partial v_z}\right],\\
t_{yz} &= \frac{1}{\tau}\left[w_x \frac{\partial W}{\partial v_x} + w_y \frac{\partial W}{\partial v_y} + (1+w_z)\frac{\partial W}{\partial v_z}\right],\\
t_{zx} &= \frac{1}{\tau}\left[(1+u_x)\frac{\partial W}{\partial w_x} + u_y \frac{\partial W}{\partial w_y} + u_z \frac{\partial W}{\partial w_z}\right],\\
t_{zy} &= \frac{1}{\tau}\left[v_x \frac{\partial W}{\partial w_x} + (1+v_y)\frac{\partial W}{\partial w_y} + v_z \frac{\partial W}{\partial w_z}\right],\\
t_{zz} &= \frac{1}{\tau}\left[w_x \frac{\partial W}{\partial w_x} + w_y \frac{\partial W}{\partial w_y} + (1+w_z)\frac{\partial W}{\partial w_z}\right].
\end{aligned}\right\} \tag{7·5}$$

These equations can also be used to determine $\partial W/\partial u_x$, $\partial W/\partial u_y$, etc., in terms of the quantities u_x, u_y, u_z, etc., if the stress-strain relationships are known.

Thus, from the first three of these equations, we have, making use of such relationships as

$$(1+u_x)\frac{\partial\tau}{\partial u_x}+v_x\frac{\partial\tau}{\partial v_x}+w_x\frac{\partial\tau}{\partial w_x}=\tau$$

and

$$u_y\frac{\partial\tau}{\partial u_x}+(1+v_y)\frac{\partial\tau}{\partial v_x}+w_y\frac{\partial\tau}{\partial w_x}=0, \tag{7·6}$$

$$\left.\begin{aligned}\frac{\partial W}{\partial u_x}&=t_{xx}\frac{\partial\tau}{\partial u_x}+t_{xy}\frac{\partial\tau}{\partial v_x}+t_{xz}\frac{\partial\tau}{\partial w_x},\\ \frac{\partial W}{\partial u_y}&=t_{xx}\frac{\partial\tau}{\partial u_y}+t_{xy}\frac{\partial\tau}{\partial v_y}+t_{xz}\frac{\partial\tau}{\partial w_y},\\ \frac{\partial W}{\partial u_z}&=t_{xx}\frac{\partial\tau}{\partial u_z}+t_{xy}\frac{\partial\tau}{\partial v_z}+t_{xz}\frac{\partial\tau}{\partial w_z}.\end{aligned}\right\} \tag{7·7}$$

From the second three equations of (7·5)

$$\left.\begin{aligned}\frac{\partial W}{\partial v_x}&=t_{yx}\frac{\partial\tau}{\partial u_x}+t_{yy}\frac{\partial\tau}{\partial v_x}+t_{yz}\frac{\partial\tau}{\partial w_x},\\ \frac{\partial W}{\partial v_y}&=t_{yx}\frac{\partial\tau}{\partial u_y}+t_{yy}\frac{\partial\tau}{\partial v_y}+t_{yz}\frac{\partial\tau}{\partial w_y},\\ \frac{\partial W}{\partial v_z}&=t_{yx}\frac{\partial\tau}{\partial u_z}+t_{yy}\frac{\partial\tau}{\partial v_z}+t_{yz}\frac{\partial\tau}{\partial w_z},\end{aligned}\right\} \tag{7·8}$$

and from the remaining three equations

$$\left.\begin{aligned}\frac{\partial W}{\partial w_x}&=t_{zx}\frac{\partial\tau}{\partial u_x}+t_{zy}\frac{\partial\tau}{\partial v_x}+t_{zz}\frac{\partial\tau}{\partial w_x},\\ \frac{\partial W}{\partial w_y}&=t_{zx}\frac{\partial\tau}{\partial u_y}+t_{zy}\frac{\partial\tau}{\partial v_y}+t_{zz}\frac{\partial\tau}{\partial w_y},\\ \frac{\partial W}{\partial w_z}&=t_{zx}\frac{\partial\tau}{\partial u_z}+t_{zy}\frac{\partial\tau}{\partial v_z}+t_{zz}\frac{\partial\tau}{\partial w_z}.\end{aligned}\right\} \tag{7·9}$$

The relations of the type $t_{yz}=t_{zy}$ impose on the possible forms that can be taken by W, the limitations

$$\left.\begin{aligned}w_x\frac{\partial W}{\partial v_x}+w_y\frac{\partial W}{\partial v_y}+(1+w_z)\frac{\partial W}{\partial v_z}&=v_x\frac{\partial W}{\partial w_x}+(1+v_y)\frac{\partial W}{\partial w_y}+v_z\frac{\partial W}{\partial w_z},\\ (1+u_x)\frac{\partial W}{\partial w_x}+u_y\frac{\partial W}{\partial w_y}+u_z\frac{\partial W}{\partial w_z}&=w_x\frac{\partial W}{\partial u_x}+w_y\frac{\partial W}{\partial u_y}+(1+w_z)\frac{\partial W}{\partial u_z}\\ \text{and}\quad v_x\frac{\partial W}{\partial u_x}+(1+v_y)\frac{\partial W}{\partial u_y}+v_z\frac{\partial W}{\partial u_z}&=(1+u_x)\frac{\partial W}{\partial v_x}+u_y\frac{\partial W}{\partial v_y}+u_z\frac{\partial W}{\partial v_z}.\end{aligned}\right\} \tag{7·10}$$

8. The stored energy function for an incompressible material

In §7 the relations between the stress components and the stored energy function are obtained on the assumption that the components δu, δv and δw of the displacements undergone by the various points of the body are independent. For an incompressible material this assumption is not valid and the incompressibility condition imposes a restriction on the allowable values of $\delta(\Delta u)$, $\delta(\Delta v)$ and $\delta(\Delta w)$.

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For an incompressible material $f(\tau) = f(1)$,

where $f(\tau)$ is an arbitrary function of τ.

Therefore, in any virtual deformation of the body,

$$\delta f(\tau) = 0, \tag{8·1}$$

at all points of the body.

Equation (8·1) is equivalent to

$$f'(1)\left[\frac{\partial \tau}{\partial u_\xi}\frac{\delta(\Delta u)}{\Delta \xi}+\frac{\partial \tau}{\partial u_\eta}\frac{\delta(\Delta u)}{\Delta \eta}+\frac{\partial \tau}{\partial u_\zeta}\frac{\delta(\Delta u)}{\Delta \zeta}+\frac{\partial \tau}{\partial v_\xi}\frac{\delta(\Delta v)}{\Delta \xi}+\frac{\partial \tau}{\partial v_\eta}\frac{\delta(\Delta v)}{\Delta \eta}+\frac{\partial \tau}{\partial v_\zeta}\frac{\delta(\Delta v)}{\Delta \zeta}\right.$$
$$\left.+\frac{\partial \tau}{\partial w_\xi}\frac{\delta(\Delta w)}{\Delta \xi}+\frac{\partial \tau}{\partial w_\eta}\frac{\delta(\Delta w)}{\Delta \eta}+\frac{\partial \tau}{\partial w_\zeta}\frac{\delta(\Delta w)}{\Delta \zeta}\right] = 0. \tag{8·2}$$

In this equation, we have written $f'(\tau) = f'(1)$, and the operators $\partial/\partial u_\xi$, $\partial/\partial u_\eta$, ... are defined as in § 7.

Thus, for an incompressible material, equation (7·4) is valid for values of $\delta(\Delta u)$, $\delta(\Delta v)$ and $\delta(\Delta w)$ for which equation (8·2) is valid and for all ratios between $\Delta\xi$, $\Delta\eta$ and $\Delta\zeta$. Consequently, the coefficients of $\delta(\Delta u)/\Delta\xi$, $\delta(\Delta u)/\Delta\eta$, ... in equations (7·4) and (8·2) are proportional. In equation (7·4), δW_1 and δW are still given by the expressions (7·2) and (7·3) respectively, but now $\tau = 1$.

Thus
$$t_{xx} = \frac{\partial W}{\partial u_\xi}+p\frac{\partial \tau}{\partial u_\xi},\quad t_{xy} = \frac{\partial W}{\partial u_\eta}+p\frac{\partial \tau}{\partial u_\eta},\quad t_{xz} = \frac{\partial W}{\partial u_\zeta}+p\frac{\partial \tau}{\partial u_\zeta},$$
$$t_{yx} = \frac{\partial W}{\partial v_\xi}+p\frac{\partial \tau}{\partial v_\xi},\quad t_{yy} = \frac{\partial W}{\partial v_\eta}+p\frac{\partial \tau}{\partial v_\eta},\quad t_{yz} = \frac{\partial W}{\partial v_\zeta}+p\frac{\partial \tau}{\partial v_\zeta},$$
$$t_{zx} = \frac{\partial W}{\partial w_\xi}+p\frac{\partial \tau}{\partial w_\xi},\quad t_{zy} = \frac{\partial W}{\partial w_\eta}+p\frac{\partial \tau}{\partial w_\eta},\quad t_{zz} = \frac{\partial W}{\partial w_\zeta}+p\frac{\partial \tau}{\partial w_\zeta},$$

where p is an arbitrary constant.

Here again we note that the relations $t_{yz} = t_{zy}$, etc., of § 2, must be obeyed.

Following out the argument of § 7, it can be seen that equations (7·5) must now be replaced by equations of the form

$$t_{xx} = (1+u_x)\left(\frac{\partial W}{\partial u_x}+p\frac{\partial \tau}{\partial u_x}\right)+u_y\left(\frac{\partial W}{\partial u_y}+p\frac{\partial \tau}{\partial u_y}\right)+u_z\left(\frac{\partial W}{\partial u_z}+p\frac{\partial \tau}{\partial u_z}\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$t_{yz} = (1+u_x)\left(\frac{\partial W}{\partial v_x}+p\frac{\partial \tau}{\partial v_x}\right)+u_y\left(\frac{\partial W}{\partial v_y}+p\frac{\partial \tau}{\partial v_y}\right)+u_z\left(\frac{\partial W}{\partial v_z}+p\frac{\partial \tau}{\partial v_z}\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

These equations may be rewritten, employing relations of the type (7·6), as

$$\left.\begin{aligned} t_{xx} &= (1+u_x)\frac{\partial W}{\partial u_x}+u_y\frac{\partial W}{\partial u_y}+u_z\frac{\partial W}{\partial u_z}+p,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ t_{yz} &= (1+u_x)\frac{\partial W}{\partial v_x}+u_y\frac{\partial W}{\partial v_y}+u_z\frac{\partial W}{\partial v_z},\\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{8·3}$$

From equations (8·3) it is quite clear that, for an incompressible material, $\partial W/\partial u_x$, $\partial W/\partial u_y$, ..., $\partial W/\partial w_z$ are given, in terms of the components of stress, by equations (7·7), (7·8) and (7·9), in which t_{xx}, t_{yy} and t_{zz} are replaced by $t_{xx}-p$, $t_{yy}-p$ and $t_{zz}-p$ respectively, giving

$$\left.\begin{aligned}\frac{\partial W}{\partial u_x} &= t_{xx}\frac{\partial \tau}{\partial u_x}+t_{xy}\frac{\partial \tau}{\partial v_x}+t_{xz}\frac{\partial \tau}{\partial w_x}-p\frac{\partial \tau}{\partial u_x},\\ \frac{\partial W}{\partial u_y} &= t_{xx}\frac{\partial \tau}{\partial u_y}+t_{xy}\frac{\partial \tau}{\partial v_y}+t_{xz}\frac{\partial \tau}{\partial w_y}-p\frac{\partial \tau}{\partial u_y},\quad \text{etc.}\end{aligned}\right\} \tag{8·4}$$

9. The stored energy function for an incompressible, neo-Hookean material

The stored energy function for an incompressible, neo-Hookean material is obtained in two ways. In the first of these, the stored energy for a general deformation is obtained by determining the stored energy for the pure, homogeneous deformation which forms part of it. In the second, the stored energy is calculated directly from the stress-strain equations (3·8), using the relationships (8·4).

Method I. Suppose we consider an element of volume of the material which, in the undeformed state, is a cube of unit edge. Suppose further that it is strained in such a manner that in the deformed state it is a cuboid whose edges are parallel to the axes of the strain ellipsoid and have lengths λ_1, λ_2 and λ_3 respectively, such that

$$\lambda_1\lambda_2\lambda_3 = 1, \tag{9·1}$$

satisfying the incompressibility condition.

Let W be the energy stored elastically in the element, in the strained state. Then, W is the work done in straining the material quasi-statically to its final state of strain. From equations (3·5) it is seen that when the dimensions of the element are $\lambda_1\times\lambda_2\times\lambda_3$, the stresses acting are given by

$$\left.\begin{aligned} t_{XX} = \tfrac{1}{3}E\lambda_1^2+p,\quad t_{YY} = \tfrac{1}{3}E\lambda_2^2+p,\\ t_{ZZ} = \tfrac{1}{3}E\lambda_3^2+p \quad\text{and}\quad t_{YZ}=t_{ZX}=t_{XY}=0.\end{aligned}\right\} \tag{9·2}$$

The element is thus subject to the action of three mutually perpendicular forces f_1, f_2 and f_3. These are given by

$$f_1 = t_{XX}\lambda_2\lambda_3,\quad f_2 = t_{YY}\lambda_3\lambda_1 \quad\text{and}\quad f_3 = t_{ZZ}\lambda_1\lambda_2,$$

since $\lambda_2\lambda_3$, $\lambda_3\lambda_1$ and $\lambda_1\lambda_2$ are the areas of the planes over which the stresses t_{XX}, t_{YY} and t_{ZZ} act.

Introducing the condition (9·1) and the formulae for t_{XX}, t_{YY} and t_{ZZ} given in equations (9·2), we have

$$f_1 = \tfrac{1}{3}E\lambda_1+\frac{p}{\lambda_1},\quad f_2 = \tfrac{1}{3}E\lambda_2+\frac{p}{\lambda_2},\quad f_3 = \tfrac{1}{3}E\lambda_3+\frac{p}{\lambda_3}.$$

The work done in straining the element of volume from dimensions $\lambda_1\times\lambda_2\times\lambda_3$ to dimensions $(\lambda_1+\delta\lambda_1)\times(\lambda_2+\delta\lambda_2)\times(\lambda_3+\delta\lambda_3)$ is

$$f_1\delta\lambda_1+f_2\delta\lambda_2+f_3\delta\lambda_3;$$

i.e.

$$\tfrac{1}{3}E(\lambda_1\delta\lambda_1+\lambda_2\delta\lambda_2+\lambda_3\delta\lambda_3)+p\left(\frac{\delta\lambda_1}{\lambda_1}+\frac{\delta\lambda_2}{\lambda_2}+\frac{\delta\lambda_3}{\lambda_3}\right).$$

Thus, the work W done, in straining the material quasi-statically from dimensions $1\times1\times1$ to dimensions $\lambda_1\times\lambda_2\times\lambda_3$, is given by

$$W=\tfrac{1}{3}E\left\{\int_1^{\lambda_1}\lambda_1\,d\lambda_1+\int_1^{\lambda_2}\lambda_2\,d\lambda_2+\int_1^{\lambda_3}\lambda_3\,d\lambda_3\right\}+p\left\{\int_1^{\lambda_1}\frac{d\lambda_1}{\lambda_1}+\int_1^{\lambda_2}\frac{d\lambda_2}{\lambda_2}+\int_1^{\lambda_3}\frac{d\lambda_3}{\lambda_3}\right\}$$
$$=\tfrac{1}{6}E(\lambda_1^2+\lambda_2^2+\lambda_3^2-3). \qquad (9\cdot3)$$

W is the stored energy per unit volume of the strained material when the principal extensions are λ_1-1, λ_2-1, λ_3-1, for as has already been seen in § 3, no work is done on the material in rotating the axes of the reciprocal strain ellipsoid into the directions of those of the strain ellipsoid.

Now, λ_1, λ_2 and λ_3 are given by the roots of the equation (1·4).

Thus $$\lambda_1^2+\lambda_2^2+\lambda_3^2=3+2(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}).$$

Whence, from equation (9·3), $$W=\tfrac{1}{3}E(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}). \qquad (9\cdot4)$$

Method II. Substituting the stress-strain relationships (3·8) for an incompressible, neo-Hookean material in equations (8·4), we find that

$$\frac{\partial W}{\partial u_x}=\tfrac{1}{3}E\Big[\{(1+u_x)^2+u_y^2+u_z^2\}\frac{\partial\tau}{\partial u_x}+\{(1+u_x)\,v_x+u_y(1+v_y)+u_zv_z\}\frac{\partial\tau}{\partial v_x}$$
$$+\{w_x(1+u_x)+w_yu_y+(1+w_z)\,u_z\}\frac{\partial\tau}{\partial w_x}\Big],$$
$$\frac{\partial W}{\partial u_y}=\tfrac{1}{3}E\Big[\{(1+u_x)^2+u_y^2+u_z^2\}\frac{\partial\tau}{\partial u_y}+\{(1+u_x)\,v_x+u_y(1+v_y)+u_zv_z\}\frac{\partial\tau}{\partial v_y}$$
$$+\{w_x(1+u_x)+w_yu_y+(1+w_z)\,u_z\}\frac{\partial\tau}{\partial w_y}\Big], \quad \text{etc.}$$

Making use of relations of the form (7·6), we obtain

$$\left.\begin{aligned}\frac{\partial W}{\partial u_x}&=\tfrac{1}{3}E(1+u_x), & \frac{\partial W}{\partial u_y}&=\tfrac{1}{3}Eu_y, & \frac{\partial W}{\partial u_z}&=\tfrac{1}{3}Eu_z,\\ \frac{\partial W}{\partial v_x}&=\tfrac{1}{3}Ev_x, & \frac{\partial W}{\partial v_y}&=\tfrac{1}{3}E(1+v_y), & \frac{\partial W}{\partial v_z}&=\tfrac{1}{3}Ev_z,\\ \frac{\partial W}{\partial w_x}&=\tfrac{1}{3}Ew_x, & \frac{\partial W}{\partial w_y}&=\tfrac{1}{3}Ew_y, & \frac{\partial W}{\partial w_z}&=\tfrac{1}{3}E(1+w_z).\end{aligned}\right\} \qquad (9\cdot5)$$

Integrating these partial differential equations, we obtain

$$W=\tfrac{1}{6}E[(1+u_x)^2+u_y^2+u_z^2+v_x^2+(1+v_y)^2+v_z^2+w_x^2+w_y^2+(1+w_z)^2]-\tfrac{1}{2}E. \qquad (9\cdot6)$$

The constant of integration is $-\tfrac{1}{2}E$, since, when $u_x=u_y=\ldots=w_z=0$, the deformation corresponds to a displacement which is possible in a rigid body and therefore $W=0$.

From equations (1·1) it can readily be seen that equation (9·6) may be written

$$W=\tfrac{1}{3}E(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}),$$

in agreement with the result (9·4) obtained by Method I.

10. Comparison of the stored energy formulae for the cases of large and small deformations

The formula for the stored energy in the case when the material is subjected to a small strain $(e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy})$ is (Love 1927, § 69)

$$2W = (\lambda+2\mu)\,(e_{xx}+e_{yy}+e_{zz})^2+\mu(e_{yz}^2+e_{zx}^2+e_{xy}^2-4e_{yy}e_{zz}-4e_{zz}e_{xx}-4e_{xx}e_{yy}), \qquad (10\cdot1)$$

where $$\mu = \frac{1}{2}\frac{E}{1+\sigma} \quad \text{and} \quad \lambda = \frac{\sigma E}{(1+\sigma)\,(1-2\sigma)}.$$

For an incompressible material equation (10·1) reduces to

$$W = \tfrac{1}{6}E(e_{yz}^2+e_{zx}^2+e_{xy}^2-4e_{yy}e_{zz}-4e_{zz}e_{xx}-4e_{xx}e_{yy}). \qquad (10\cdot2)$$

This equation does not at first sight appear to be in accord with equation (9·4). However, it will be shown that equation (9·4) does in fact reduce to equation (10·2) when the strains are small.

It is seen in § 9 that equation (9·4) is equivalent to equation (9·3). Writing $\lambda_1 = 1+e_1$, $\lambda_2 = 1+e_2$, $\lambda_3 = 1+e_3$, where e_1, e_2 and e_3 are the principal extensions for a small strain, we have

$$W = \tfrac{1}{6}E[2(e_1+e_2+e_3)+(e_1+e_2+e_3)^2-2(e_1e_2+e_2e_3+e_3e_1)]. \qquad (10\cdot3)$$

e_1, e_2, e_3 are the roots (Love 1927, § 11) of the equation

$$\begin{vmatrix} e_{xx}-e & \tfrac{1}{2}e_{xy} & \tfrac{1}{2}e_{zx} \\ \tfrac{1}{2}e_{xy} & e_{yy}-e & \tfrac{1}{2}e_{yz} \\ \tfrac{1}{2}e_{zx} & \tfrac{1}{2}e_{yz} & e_{zz}-e \end{vmatrix} = 0.$$

Now $$e_1+e_2+e_3 = -(e_1e_2+e_2e_3+e_3e_1)$$

for an incompressible material, and

$$e_1e_2+e_2e_3+e_3e_1 = e_{xx}e_{yy}+e_{yy}e_{zz}+e_{zz}e_{xx}-\tfrac{1}{4}(e_{yz}^2+e_{zx}^2+e_{xy}^2).$$

Introducing these results into equation (10·3) and neglecting second degree terms in $(e_1+e_2+e_3)$, we obtain

$$W = \tfrac{1}{6}E(e_{yz}^2+e_{zx}^2+e_{xy}^2-4e_{yy}e_{zz}-4e_{zz}e_{xx}-4e_{xx}e_{yy}),$$

thus proving that the formulae (9·4) and (10·2), for the stored energy in the cases of large and small strains of an incompressible material, are consistent.

11. Stress-strain relationships of ideal and real rubbers

The kinetic theory of the high-elasticity of rubber-like materials, which was originally put forward by Meyer, von Susich & Valko (1932), has been developed to a considerable extent by later workers. In this theory, an ideal rubber-like material is considered. This consists of a number of high polymeric chains, permanently cross-linked at a certain number of junction-points along their length, to form a network. The segment of any chain between two consecutive junction-points along the chain possesses a large number of rotational degrees of freedom, so that if the two junction-points are considered fixed, the segment can take up a large number of possible geometrical configurations under the influence of its thermal motion. The entropy of a segment, for any specified positions of its terminating junction-points, is considered to be proportional to the logarithm of the volume of the rotation-configurational phase-space which it fills. The stored elastic energy of the deformed rubber arises from the fact that the average entropy of the segments in the deformed state is smaller than that in the undeformed state. A review of the general physical basis of this theory has been given by Treloar (1943*a*) and, more recently, by Flory (1944).

In applying this model of the mechanism of high-elasticity, Wall (1942*a*, *b*) has assumed that all the chain segments are of equal lengths and that the material is incompressible.

The assumption of incompressibility will be justified if the strain produced by a pure hydrostatic pressure is small compared with that produced by a normal or tangential traction of equal magnitude. Measurements on the compressibility of rubbers, by Holt & McPherson (1937), bear out this assumption. It will, of course, be generally true for any highly elastic material which is neither cellular nor porous. Certain further plausible assumptions are also made in order to make the problem tractable to calculation. Flory & Rehner (1943) also make the assumptions of uniformity of segment length and incompressibility, but make somewhat different simplifying assumptions from Wall's in carrying out the analysis. In both the analyses of Wall and of Flory & Rehner, the assumption is made that the chain segments are not nearly fully extended.

Treloar (1943*b*, 1946) has used both the methods of Wall and of Flory & Rehner to obtain the stored energy function for an ideal rubber-like material subject to a general, pure, homogeneous strain and has shown that both methods lead to the same result. This is

$$W = \tfrac{1}{2}NkT(\lambda_1^2+\lambda_2^2+\lambda_3^2-3), \tag{11·1}$$

where N is the number of segments per unit volume, k is Boltzmann's constant and T is the absolute temperature. This formula has the same form as (9·3), which gives the stored energy function for an incompressible, neo-Hookean material, and we reach the important conclusion that the ideal rubber-like material, as conceived by Wall, Flory & Rehner and Treloar, is an incompressible, neo-Hookean material. For the ideal rubber-like material the elastic modulus E is given by $3NkT$. However, from the nature of the simplifying assumptions made in the statistical treatment, it is not to be expected that the quantity $3NkT$ has any exact significance in practice. In comparing equation (11·1) with experimental results on real rubbers, it is to be expected that, at most, the shapes of experimental and theoretical stress-strain curves will be similar.

The experimentally obtained load-elongation curves for certain vulcanized natural rubbers, under simple elongation and under pure shear, have been compared by Treloar (1944) with the appropriate formulae obtained theoretically as special cases of (11·1). It was found that the agreement was moderately good up to fairly high strains. At extreme values of the strain, when the chain segments are approaching full extension, the theoretical and experimental curves depart considerably from each other. This is to be expected in view of the assumption, which is made in deducing (11·1), that the chain segments are not nearly fully extended.

Gee (1946) has compared the formula (11·1) with experimentally obtained data for both swollen and unswollen, vulcanized rubbers, in simple extension. He has found that the agreement becomes more and more exact, over the range of elongations where the chain segments are not nearly fully extended, as the amount of the swelling agent is increased. The detailed explanation of these results is not yet clear, but in general it is to be expected that the assumptions, made in the statistical-mechanical derivation of equation (11·1), will be more accurately valid in a swollen rubber than in an unswollen one. For, the van der Waals forces between the chain segments and the phenomenon of crystallization under strain, which exist in a real rubber, are not taken into account in the statistical-mechanical treatment of an ideal rubber-like material and will, in general, be reduced by swelling.

It appears from the remarks made in this section that there is considerable justification for developing the mechanics of rubber-like materials on the assumption that they are incompressible, neo-Hookean materials. However, the departure of experimental results, from theoretical results so obtained, might be expected to be large when the rubber is in a condition of very large strain or when crystallization has taken place. Also, considerable divergences are to be expected in the case of dynamic conditions, where the van der Waals forces between the segments may have an important effect.

PART B. THE EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

12. Introduction

Two methods have been used for deducing the equations of motion and the boundary conditions for an elastic body, which suffers small strains under the action of body forces (X, Y, Z) per unit mass and surface forces (X_ν, Y_ν, Z_ν) per unit area of surface, parallel to the axes (x, y, z) of a fixed rectangular, Cartesian, co-ordinate system.

In the first of these methods, the forces acting on an elementary cuboid of the strained material, with its edges parallel to the co-ordinate axes, are resolved to give the equations of motion in the form (Love 1927, § 54)

$$\rho\frac{\partial^2 u}{\partial t^2} = \rho X + \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z}, \quad \text{etc.,} \tag{12·1}$$

where ρ is the density of the material.

The boundary conditions are obtained from the consideration that the components of stress at the surface must be in equilibrium with the applied surface forces (Love 1927, § 47).

In the second method, due to Kirchhoff (Love 1927, § 115), the equations of motion are deduced from the Hamiltonian principle, by the methods of the Calculus of Variations. They take the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \rho X + \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial e_{xx}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial e_{xy}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial e_{xz}}\right), \quad \text{etc.} \tag{12·2}$$

Here W is the elastically stored energy per unit volume of the material, measured in the undeformed condition, at the time t and $(e_{xx}, e_{yy}, \ldots)$ are the components of strain defined by equations (1·2).

The boundary conditions are also obtained from the Hamiltonian Principle in the same analysis.

For large strains, neither equations (12·1) or (12·2) apply. However, analogues of both methods can be used to deduce the appropriate equations of motion and boundary conditions.

13. The equations of motion

Let us consider the motion of an elementary cuboid of the strained material, situated at $(\xi, \eta, \zeta) = (x+u, y+v, z+w)$ and having its edges, of length $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, parallel to the co-ordinate axes x, y, z respectively.

The difference between the forces, due to the stress in the material, acting on opposite faces of the cuboid, parallel to the x-axis, is

$$\left[\frac{\partial t_{xx}}{\partial \xi} + \frac{\partial t_{xy}}{\partial \eta} + \frac{\partial t_{xz}}{\partial \zeta}\right] \Delta\xi \Delta\eta \Delta\zeta.$$

The component of the body forces, acting on the cuboid, parallel to the x-axis is $(1/\tau)\,\rho\Delta\xi\Delta\eta\Delta\zeta X$, where ρ is the density of the material in the undeformed state, so that the mass of the cuboid is $(1/\tau)\,\rho\Delta\xi\Delta\eta\Delta\zeta$.

Applying Newton's Second Law to the component of motion of this cuboid in a direction parallel to the x-axis, we have

$$\frac{1}{\tau}\rho\Delta\xi\Delta\eta\Delta\zeta\frac{\partial^2\xi}{\partial t^2} = \frac{1}{\tau}\rho\Delta\xi\Delta\eta\Delta\zeta X + \left[\frac{\partial t_{xx}}{\partial \xi} + \frac{\partial t_{xy}}{\partial \eta} + \frac{\partial t_{xz}}{\partial \zeta}\right] \Delta\xi\Delta\eta\Delta\zeta. \tag{13·1}$$

This yields the first of the equations of motion:

$$\left.\begin{aligned}
\frac{\rho}{\tau}\frac{\partial^2 u}{\partial t^2} &= \frac{\rho}{\tau}X + \left[\frac{\partial t_{xx}}{\partial \xi} + \frac{\partial t_{xy}}{\partial \eta} + \frac{\partial t_{xz}}{\partial \zeta}\right],\\
\frac{\rho}{\tau}\frac{\partial^2 v}{\partial t^2} &= \frac{\rho}{\tau}Y + \left[\frac{\partial t_{yx}}{\partial \xi} + \frac{\partial t_{yy}}{\partial \eta} + \frac{\partial t_{yz}}{\partial \zeta}\right],\\
\frac{\rho}{\tau}\frac{\partial^2 w}{\partial t^2} &= \frac{\rho}{\tau}Z + \left[\frac{\partial t_{zx}}{\partial \xi} + \frac{\partial t_{zy}}{\partial \eta} + \frac{\partial t_{zz}}{\partial \zeta}\right].
\end{aligned}\right\} \tag{13·2}$$

The second and third of equations (13·2) can be obtained in a manner similar to the first, by applying Newton's Second Law to the motion of the cuboid parallel to the y and z axes respectively.

14. The boundary conditions

At the boundary, the x, y and z components of the stress balance the applied surface forces. Thus, let (X_ν, Y_ν, Z_ν) be the components, parallel to the co-ordinate axes, of the surface forces, per unit area of the surface measured in the undeformed state. The corresponding components of the surface forces, per unit area of the surface, measured in the deformed state, are $X_\nu(dS/dS')$, $Y_\nu(dS/dS')$ and $Z_\nu(dS/dS')$, where dS' is the area in the deformed state of an element of the surface, which has an area dS in the undeformed state.

Equating these components of the applied surface forces to the corresponding components of the stress, we have the boundary conditions at the surface

$$\left.\begin{aligned}
X_\nu\frac{dS}{dS'} &= t_{xx}\cos(x,\nu') + t_{xy}\cos(y,\nu') + t_{xz}\cos(z,\nu'),\\
Y_\nu\frac{dS}{dS'} &= t_{yx}\cos(x,\nu') + t_{yy}\cos(y,\nu') + t_{yz}\cos(z,\nu'),\\
Z_\nu\frac{dS}{dS'} &= t_{zx}\cos(x,\nu') + t_{zy}\cos(y,\nu') + t_{zz}\cos(z,\nu'),
\end{aligned}\right\} \tag{14·1}$$

where $\cos(x,\nu')$, $\cos(y,\nu')$, $\cos(z,\nu')$ are the direction-cosines of the normal to the surface in the strained state.

15. The variational equations of motion for large deformations

W is considered to be a function of the nine partial derivatives of the displacement components $(u_x, u_y, u_z, v_x, v_y, v_z, w_x, w_y, w_z)$.

Now, let T be the total kinetic energy of motion of the body at any instant t, let V be the total energy stored elastically in the body and let δW_1 be the work done by the externally applied body and surface forces when the displacement configuration is varied by a small amount. It follows from the Hamiltonian principle that

$$\delta\int_{t_0}^{t_1}(T-V)\,dt+\int_{t_0}^{t_1}\delta W_1\,dt=0, \tag{15·1}$$

where the operator δ denotes any small variation of the displacement configuration from that obtaining at the instant of time t. It is assumed that the variations $(\delta u, \delta v, \delta w)$ of the displacement components (u, v, w) are zero throughout the body at t_0 and t_1. The integration is carried out between any two arbitrary times t_0 and t_1. Now

$$T=\int\tfrac{1}{2}\rho\left\{\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right\}d\tau.$$

It follows (Love 1927, § 115) that

$$\delta\int_{t_0}^{t_1}T\,dt=-\int_{t_0}^{t_1}dt\int\rho\left(\frac{\partial^2 u}{\partial t^2}\,\delta u+\frac{\partial^2 v}{\partial t^2}\,\delta v+\frac{\partial^2 w}{\partial t^2}\,\delta w\right)d\tau, \tag{15·2}$$

bearing in mind that $(\delta u, \delta v, \delta w)$ are zero at $t=t_0$ and $t=t_1$. The volume integration is carried out over the whole volume of the material in the undeformed state.

Let (X, Y, Z) denote the components of the body forces per unit mass of the material, situated at (x, y, z) in the undeformed state of the material. Let $(X_{\nu'}, Y_{\nu'}, Z_{\nu'})$ be the components of the surface forces, per unit area of the surface in the deformed state. ν' denotes the direction of the normal to the deformed surface. Then

$$\delta W_1=\int\rho(X\,\delta u+Y\,\delta v+Z\,\delta w)\,d\tau+\int(X_{\nu'}\,\delta u+Y_{\nu'}\,\delta v+Z_{\nu'}\,\delta w)\,dS'. \tag{15·3}$$

Here again the volume integration is carried out over the whole volume of the material in the undeformed state, but the surface integration is carried out over the surface of the deformed material, dS' denoting an element of area of this deformed surface.

We may rewrite equation (15·3)

$$\delta W_1=\int\rho(X\,\delta u+Y\,\delta v+Z\,\delta w)\,d\tau+\int(X_{\nu'}\,\delta u+Y_{\nu'}\,\delta v+Z_{\nu'}\,\delta w)\,\frac{dS'}{dS}\,dS, \tag{15·4}$$

where dS is the area of an element of the undeformed surface corresponding to the element dS' of the deformed surface. The surface integration is now carried out over the whole of the surface of the undeformed body.

The elastically stored energy $V=\int W\,d\tau$. Bearing in mind that W is a function of $(u_x, u_y, u_z, \ldots)$, we have

$$\delta V = \int\left\{\frac{\partial W}{\partial u_x}\frac{\partial}{\partial x}(\delta u)+\frac{\partial W}{\partial u_y}\frac{\partial}{\partial y}(\delta u)+\dots\right\}d\tau$$

$$= \int\left[\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial u_x}\delta u\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial u_y}\delta u\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial u_z}\delta u\right\}+\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial v_x}\delta v\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial v_y}\delta v\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial v_z}\delta v\right\}\right.$$

$$\left.+\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial w_x}\delta w\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial w_y}\delta w\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial w_z}\delta w\right\}\right]d\tau-\int\left[\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial u_x}\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial u_y}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial u_z}\right\}\right]\delta u\,d\tau$$

$$-\int\left[\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial v_x}\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial v_y}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial v_z}\right\}\right]\delta v\,d\tau-\int\left[\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial w_x}\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial w_y}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial w_z}\right\}\right]\delta w\,d\tau. \quad (15\cdot5)$$

Applying Green's theorem, we find that

$$\int\left[\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial u_x}\delta u\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial u_y}\delta u\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial u_z}\delta u\right\}+\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial v_x}\delta v\right\}+\dots\right]d\tau$$

$$=\int\left[\frac{\partial W}{\partial u_x}\cos(x,\nu)+\frac{\partial W}{\partial u_y}\cos(y,\nu)+\frac{\partial W}{\partial u_z}\cos(z,\nu)\right]\delta u\,dS$$

$$+\int\left[\frac{\partial W}{\partial v_x}\cos(x,\nu)+\frac{\partial W}{\partial v_y}\cos(y,\nu)+\frac{\partial W}{\partial v_z}\cos(z,\nu)\right]\delta v\,dS$$

$$+\int\left[\frac{\partial W}{\partial w_x}\cos(x,\nu)+\frac{\partial W}{\partial w_y}\cos(y,\nu)+\frac{\partial W}{\partial w_z}\cos(z,\nu)\right]\delta w\,dS,$$

where (x,ν), (y,ν) and (z,ν) denote the angles between the normal to the surface in the undeformed state, at the element dS, and the co-ordinate axes x, y and z respectively.

Introducing this result into equation (15·5) and substituting in (15·1) for $\delta\int_{t_0}^{t_1}T\,dt$ from equation (15·2), for δW_1 from equation (15·4) and for δV from equation (15·5), we have

$$-\int_{t_0}^{t_1}dt\int\rho\left(\frac{\partial^2 u}{\partial t^2}\delta u+\frac{\partial^2 v}{\partial t^2}\delta v+\frac{\partial^2 w}{\partial t^2}\delta w\right)d\tau$$

$$-\int_{t_0}^{t_1}dt\int\left[\frac{\partial W}{\partial u_x}\cos(x,\nu)+\frac{\partial W}{\partial u_y}\cos(y,\nu)+\frac{\partial W}{\partial u_z}\cos(z,\nu)\right]\delta u\,dS$$

$$-\int_{t_0}^{t_1}dt\int\left[\frac{\partial W}{\partial v_x}\cos(x,\nu)+\frac{\partial W}{\partial v_y}\cos(y,\nu)+\frac{\partial W}{\partial v_z}\cos(z,\nu)\right]\delta v\,dS$$

$$-\int_{t_0}^{t_1}dt\int\left[\frac{\partial W}{\partial w_x}\cos(x,\nu)+\frac{\partial W}{\partial w_y}\cos(y,\nu)+\frac{\partial W}{\partial w_z}\cos(z,\nu)\right]\delta w\,dS$$

$$+\int_{t_0}^{t_1}dt\int\left[\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial u_x}\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial u_y}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial u_z}\right\}\right]\delta u\,d\tau$$

$$+\int_{t_0}^{t_1}dt\int\left[\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial v_x}\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial v_y}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial v_z}\right\}\right]\delta v\,d\tau$$

$$+\int_{t_0}^{t_1}dt\int\left[\frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial w_x}\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial w_y}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial w_z}\right\}\right]\delta w\,d\tau$$

$$+\int_{t_0}^{t_1}dt\int\rho(X\delta u+Y\delta v+Z\delta w)\,d\tau$$

$$+\int_{t_0}^{t_1}dt\int(X_{\nu'}\delta u+Y_{\nu'}\delta v+Z_{\nu'}\delta w)\frac{dS'}{dS}dS=0. \quad (15\cdot6)$$

The components $(\delta u, \delta v, \delta w)$ of the infinitesimally small virtual displacements at each point may be chosen arbitrarily, subject only to the conditions that they are continuous functions of the position of the point and of time and they are consistent with geometrical considerations. Also, we have already assumed in deriving equation (15·6) that

$$\delta u = \delta v = \delta w = 0,$$

when $t = t_0$ and $t = t_1$. Since the limits t_0 and t_1 may be arbitrarily chosen, let us choose them to be separated by an infinitesimally small time interval Δt, during which time the integrands are substantially constant. In this time interval, suppose that $\delta v = \delta w = 0$, at all points of the body, and $\delta u = 0$, at all points except over a small element which, in the undeformed state, has volume $\Delta x \times \Delta y \times \Delta z$ and is situated at (x, y, z). In this element δu has a constant value, infinitesimally small compared with $\Delta(x+u)$, $\Delta(y+v)$ and $\Delta(z+w)$, except in an infinitesimally small region at the surfaces of the element of thickness greater than δu. In this region δu falls to zero. Also, δu, δv and δw are constant throughout the time interval Δt, except in infinitesimal intervals of the time, small compared with Δt, at t_0 and t_1. In the small intervals of time and volume considered, the coefficients of δu in the integrands of equation (15·6) may be considered constant, giving the first of the three equations of motion

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= \rho X + \frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial u_x}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial u_y}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial u_z}\right\}, \\ \rho\frac{\partial^2 v}{\partial t^2} &= \rho Y + \frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial v_x}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial v_y}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial v_z}\right\} \\ \text{and}\quad \rho\frac{\partial^2 w}{\partial t^2} &= \rho Z + \frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial w_x}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial w_y}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial w_z}\right\}. \end{aligned}\right\} \tag{15·7}$$

The second and third of these equations are obtained in a manner analogous to the first.

16. Alternative form of the boundary conditions

The boundary conditions may also be obtained from equation (15·6). Thus, we take $\delta v = \delta w = 0$ at all points of the body and $\delta u = 0$ at all points, except over a vanishingly small area of the surface. Again, δu is constant over this area and infinitesimally small compared with the linear dimensions of the area. It falls to zero in a distance at the edges of the area greater than, but comparable with, the constant value of δu and similarly in a direction normal to the surface. The variation with time in the interval Δt $(= t_1 - t_0)$ is similar to that described in the previous section. This gives

$$\frac{dS'}{dS}X_{\nu'} = X_\nu = \frac{\partial W}{\partial u_x}\cos(x,\nu) + \frac{\partial W}{\partial u_y}\cos(y,\nu) + \frac{\partial W}{\partial u_z}\cos(z,\nu).$$

Two further boundary conditions may be similarly derived, giving the three boundary conditions

$$\left.\begin{aligned} X_\nu &= \frac{\partial W}{\partial u_x}\cos(x,\nu) + \frac{\partial W}{\partial u_y}\cos(y,\nu) + \frac{\partial W}{\partial u_z}\cos(z,\nu), \\ Y_\nu &= \frac{\partial W}{\partial v_x}\cos(x,\nu) + \frac{\partial W}{\partial v_y}\cos(y,\nu) + \frac{\partial W}{\partial v_z}\cos(z,\nu), \\ Z_\nu &= \frac{\partial W}{\partial w_x}\cos(x,\nu) + \frac{\partial W}{\partial w_y}\cos(y,\nu) + \frac{\partial W}{\partial w_z}\cos(z,\nu). \end{aligned}\right\} \tag{16·1}$$

For a free surface, we put $X_\nu = Y_\nu = Z_\nu = 0$ in these equations.

If the surface has the equation $\qquad f(x,y,z) = 0,$

we note that the direction cosines of the normal to the surface at (x,y,z) are proportional to (f_x, f_y, f_z). Substituting these for $\cos(x,\nu)$, $\cos(y,\nu)$ and $\cos(z,\nu)$, the boundary conditions for the free surface become

$$\left.\begin{aligned} \frac{\partial W}{\partial u_x}f_x+\frac{\partial W}{\partial u_y}f_y+\frac{\partial W}{\partial u_z}f_z &= 0,\\ \frac{\partial W}{\partial v_x}f_x+\frac{\partial W}{\partial v_y}f_y+\frac{\partial W}{\partial v_z}f_z &= 0\\ \text{and}\qquad \frac{\partial W}{\partial w_x}f_x+\frac{\partial W}{\partial w_y}f_y+\frac{\partial W}{\partial w_z}f_z &= 0. \end{aligned}\right\} \tag{16·2}$$

17. Correlation of the equations of motion

In § 13, the equations of motion (13·2), of a highly elastic material, are obtained from the consideration of the forces acting on an element of the strained material. In § 15, the equations of motion (15·7) are obtained from the Hamiltonian Principle. It can be shown by the use of the expressions (7·7), (7·8) and (7·9) relating the stress components and the stored energy that these two sets of equations are identical.

Thus, consider the equation

$$\rho\frac{\partial^2 u}{\partial t^2} = \rho X + \frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial u_x}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial u_y}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial u_z}\right\}.$$

Substituting in this the expressions for $\partial W/\partial u_x$, $\partial W/\partial u_y$ and $\partial W/\partial u_z$, given in equations (7·7), it is seen that this equation may be rewritten

$$\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} - \rho X &= \frac{\partial}{\partial x}\left[t_{xx}\frac{\partial \tau}{\partial u_x}+t_{xy}\frac{\partial \tau}{\partial v_x}+t_{xz}\frac{\partial \tau}{\partial w_x}\right] + \frac{\partial}{\partial y}\left[t_{xx}\frac{\partial \tau}{\partial u_y}+t_{xy}\frac{\partial \tau}{\partial v_y}+t_{xz}\frac{\partial \tau}{\partial w_y}\right]\\ &\quad + \frac{\partial}{\partial z}\left[t_{xx}\frac{\partial \tau}{\partial u_z}+t_{xy}\frac{\partial \tau}{\partial v_z}+t_{xz}\frac{\partial \tau}{\partial w_z}\right]\\ &= \frac{\partial \tau}{\partial u_x}\frac{\partial t_{xx}}{\partial x}+\frac{\partial \tau}{\partial u_y}\frac{\partial t_{xx}}{\partial y}+\frac{\partial \tau}{\partial u_z}\frac{\partial t_{xx}}{\partial z}+\frac{\partial \tau}{\partial v_x}\frac{\partial t_{xy}}{\partial x}+\frac{\partial \tau}{\partial v_y}\frac{\partial t_{xy}}{\partial y}+\frac{\partial \tau}{\partial v_z}\frac{\partial t_{xy}}{\partial z}\\ &\quad +\frac{\partial \tau}{\partial w_x}\frac{\partial t_{xz}}{\partial x}+\frac{\partial \tau}{\partial w_y}\frac{\partial t_{xz}}{\partial y}+\frac{\partial \tau}{\partial w_z}\frac{\partial t_{xz}}{\partial z}+t_{xx}\left[\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial u_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial u_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial u_z}\right)\right]\\ &\quad +t_{xy}\left[\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial v_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial v_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial v_z}\right)\right]+t_{xz}\left[\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial w_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial w_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial w_z}\right)\right]. \end{aligned} \tag{17·1}$$

Now,

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial u_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial u_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial u_z}\right)\equiv 0,\\ &\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial v_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial v_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial v_z}\right)\equiv 0\\ \text{and}\qquad &\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial w_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial w_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial w_z}\right)\equiv 0. \end{aligned}$$

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Also, introducing the operational relationships

$$\left.\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}\\ &= (1+u_x)\frac{\partial}{\partial \xi}+v_x\frac{\partial}{\partial \eta}+w_x\frac{\partial}{\partial \zeta},\\ \frac{\partial}{\partial y} &= \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial y}\frac{\partial}{\partial \zeta}\\ &= u_y\frac{\partial}{\partial \xi}+(1+v_y)\frac{\partial}{\partial \eta}+w_y\frac{\partial}{\partial \zeta}\\ \frac{\partial}{\partial z} &= \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}+\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}+\frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\\ &= u_z\frac{\partial}{\partial \xi}+v_z\frac{\partial}{\partial \eta}+(1+w_z)\frac{\partial}{\partial \zeta},\end{aligned}\right\} \tag{17·2}$$

and

equation (17·1) becomes

$$\begin{aligned}\rho\frac{\partial^2 u}{\partial t^2} = \rho X &+\left[(1+u_x)\frac{\partial \tau}{\partial u_x}+u_y\frac{\partial \tau}{\partial u_y}+u_z\frac{\partial \tau}{\partial u_z}\right]\frac{\partial t_{xx}}{\partial \xi}+\left[v_x\frac{\partial \tau}{\partial u_x}+(1+v_y)\frac{\partial \tau}{\partial u_y}+v_z\frac{\partial \tau}{\partial u_z}\right]\frac{\partial t_{xx}}{\partial \eta}\\ &+\left[w_x\frac{\partial \tau}{\partial u_x}+w_y\frac{\partial \tau}{\partial u_y}+(1+w_z)\frac{\partial \tau}{\partial u_z}\right]\frac{\partial t_{xx}}{\partial \zeta}+\left[(1+u_x)\frac{\partial \tau}{\partial v_x}+u_y\frac{\partial \tau}{\partial v_y}+u_z\frac{\partial \tau}{\partial v_z}\right]\frac{\partial t_{xy}}{\partial \xi}\\ &+\left[v_x\frac{\partial \tau}{\partial v_x}+(1+v_y)\frac{\partial \tau}{\partial v_y}+v_z\frac{\partial \tau}{\partial v_z}\right]\frac{\partial t_{xy}}{\partial \eta}+\left[w_x\frac{\partial \tau}{\partial v_x}+w_y\frac{\partial \tau}{\partial v_y}+(1+w_z)\frac{\partial \tau}{\partial v_z}\right]\frac{\partial t_{xy}}{\partial \zeta}\\ &+\left[(1+u_x)\frac{\partial \tau}{\partial w_x}+u_y\frac{\partial \tau}{\partial w_y}+u_z\frac{\partial \tau}{\partial w_z}\right]\frac{\partial t_{xz}}{\partial \xi}+\left[v_x\frac{\partial \tau}{\partial w_x}+(1+v_y)\frac{\partial \tau}{\partial w_y}+v_z\frac{\partial \tau}{\partial w_z}\right]\frac{\partial t_{xz}}{\partial \eta}\\ &+\left[w_x\frac{\partial \tau}{\partial w_x}+w_y\frac{\partial \tau}{\partial w_y}+(1+w_z)\frac{\partial \tau}{\partial w_z}\right]\frac{\partial t_{xz}}{\partial \zeta}.\end{aligned} \tag{17·3}$$

Now, noting relationships of the type (7·6), equation (17·3) becomes

$$\frac{\rho}{\tau}\frac{\partial^2 u}{\partial t^2} = \frac{\rho}{\tau}X+\frac{\partial t_{xx}}{\partial \xi}+\frac{\partial t_{xy}}{\partial \eta}+\frac{\partial t_{xz}}{\partial \zeta},$$

which is the first of the equations of motion (13·2).

Using a similar method, it can be shown that the remaining two pairs of equations of motion are in agreement.

18. Correlation of the boundary conditions

Equations (7·7), (7·8) and (7·9) can also be used to show that the two forms (14·1) and (16·1) for the boundary conditions are in agreement.

Thus, consider the equation

$$X_\nu = \frac{\partial W}{\partial u_x}\cos(x,\nu)+\frac{\partial W}{\partial u_y}\cos(y,\nu)+\frac{\partial W}{\partial u_z}\cos(z,\nu). \tag{18·1}$$

$\cos(x,\nu)$, $\cos(y,\nu)$ and $\cos(z,\nu)$ can be expressed in terms of $\cos(x,\nu')$, $\cos(y,\nu')$ and $\cos(z,\nu')$ in the following manner.

Let $f(x,y,z) = 0$ be the equation of the surface of the body in its undeformed state.

Since (ξ, η, ζ) are the co-ordinates of a point in the deformed material, which lies at (x, y, z) in the undeformed state, the equation of the deformed surface is

$$f(\xi-u, \eta-v, \zeta-w) = 0.$$

The direction-cosines $\cos(x,\nu)$, $\cos(y,\nu)$, $\cos(z,\nu)$, of the normal to the undeformed surface at (x,y,z), are

$$(f_x^2+f_y^2+f_z^2)^{-\frac{1}{2}}(f_x, f_y, f_z).$$

The direction-cosines $\cos(x,\nu')$, $\cos(y,\nu')$, $\cos(z,\nu')$, of the normal at the corresponding point of the deformed surface, are

$$(f_\xi^2+f_\eta^2+f_\zeta^2)^{-\frac{1}{2}}(f_\xi, f_\eta, f_\zeta).$$

Now
$$\frac{f_x^2+f_y^2+f_z^2}{f_\xi^2+f_\eta^2+f_\zeta^2} = \frac{dv'}{dv}, \tag{18·2}$$

where dv is the length of the perpendicular from the point $(x-dx, y-dy, z-dz)$ to the tangent plane to the surface of the undeformed body at (x,y,z) and dv' is that from the point $(\xi-d\xi, \eta-d\eta, \zeta-d\zeta)$ on to the tangent plane to the surface of the deformed body at (ξ,η,ζ).

Employing the relations (17·2) and (18·2), we obtain

$$\cos(x,\nu)\, dv'/dv = (1+u_x)\cos(x,\nu') + v_x \cos(y,\nu') + w_x \cos(z,\nu'),$$
$$\cos(y,\nu)\, dv'/dv = u_y \cos(x,\nu') + (1+v_y)\cos(y,\nu') + w_y \cos(z,\nu')$$
and
$$\cos(z,\nu)\, dv'/dv = u_z \cos(x,\nu') + v_z \cos(y,\nu') + (1+w_z)\cos(z,\nu'). \tag{18·3}$$

Substituting in equation (18·1) for $\partial W/\partial u_x$, $\partial W/\partial u_y$ and $\partial W/\partial u_z$, from equations (7·7), and for $\cos(x,\nu)$, $\cos(y,\nu)$ and $\cos(z,\nu)$, from equations (18·3), and making use of relations of the types (7·6), we obtain

$$X_\nu = (dv/dv')\,\tau[t_{xx}\cos(x,\nu') + t_{xy}\cos(y,\nu') + t_{xz}\cos(z,\nu')]. \tag{18·4}$$

Now,
$$\frac{dS'\,dv'}{dS\,dv} = \tau.$$

So, equation (18·4) becomes

$$X_\nu\, dS/dS' = t_{xx}\cos(x,\nu') + t_{xy}\cos(y,\nu') + t_{xz}\cos(z,\nu'),$$

which is in agreement with the first of the three boundary conditions given in equations (14·1).

In a similar manner it can be shown that the remaining two boundary conditions given in equations (16·1) are equivalent to the remaining two boundary conditions given in equations (14·1).

19. Equations of motion and boundary conditions for an incompressible material

In deducing the equations of motion and boundary conditions for an incompressible material, we must again take account of the fact discussed in § 8, that the allowable choice of the components δu, δv, δw of the virtual displacements is restricted by the incompressibility.

If $\delta\tau$ is a small volume element, measured in the undeformed state, we have

$$f(\tau)\,\delta\tau = f(1)\,\delta\tau,$$

where $f(\tau)$ is an arbitrary function of τ, which may vary in form from point to point of the material in a continuous manner.

Integrating throughout the whole volume of the body in the undeformed state,

$$\int f(\tau)\,d\tau = \int f(1)\,d\tau.$$

The components δu, δv and δw are thus restricted by the relation

$$\delta\int f(\tau)\,d\tau = 0,$$

at any instant of time.

Since this relation is valid at all instants of time, it follows that

$$\int_{t_0}^{t_1} dt \int \delta f(\tau)\,d\tau = 0. \tag{19·1}$$

For an incompressible material, the variational equation (15·6) is thus subject to the restricting condition of equation (19·1). Therefore, the equation

$$\delta\int_{t_0}^{t_1}(T-V)\,dt + \int_{t_0}^{t_1}\delta W_1\,dt + \int_{t_0}^{t_1} dt\int \delta[\lambda f(\tau)]\,d\tau = 0 \tag{19·2}$$

must be valid, where λ is an arbitrary constant, for all variations of δu, δv and δw possible in a compressible material. Now, since $V = \int W\,d\tau$, we can obtain equation (19·2), by substituting $W-\lambda f(\tau)$ for W in equation (15·1).

Therefore, the equations of motion for an incompressible material can be obtained by substituting $W-\lambda f(\tau)$ for W in equations (15·7), etc. Writing $-p = \lambda f'(\tau)$, we obtain

$$\rho\frac{\partial^2 u}{\partial t^2} = \rho X + \frac{\partial}{\partial x}\left\{\frac{\partial W}{\partial u_x} + p\frac{\partial \tau}{\partial u_x}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial W}{\partial u_y} + p\frac{\partial \tau}{\partial u_y}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial W}{\partial u_z} + p\frac{\partial \tau}{\partial u_z}\right\}, \tag{19·3}$$

and two similar equations in which u and X are replaced by v and Y and by w and Z respectively.

Since
$$\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial u_z}\right) = 0,$$

equation (19·3) becomes

$$\rho\frac{\partial^2 u}{\partial t^2} = \rho X + \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial u_z}\right) + \frac{\partial \tau}{\partial u_x}\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial u_y}\frac{\partial p}{\partial y} + \frac{\partial \tau}{\partial u_z}\frac{\partial p}{\partial z}. \tag{19·4}$$

In a similar manner the two other equations of motion may be obtained from the remaining two equations of (15·9), giving, together with (19·4), the three equations of motion for an incompressible material:

$$\left.\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} &= \rho X + \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial u_z}\right) + \frac{\partial \tau}{\partial u_x}\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial u_y}\frac{\partial p}{\partial y} + \frac{\partial \tau}{\partial u_z}\frac{\partial p}{\partial z},\\
\rho\frac{\partial^2 v}{\partial t^2} &= \rho Y + \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial v_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial v_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial v_z}\right) + \frac{\partial \tau}{\partial v_x}\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial v_y}\frac{\partial p}{\partial y} + \frac{\partial \tau}{\partial v_z}\frac{\partial p}{\partial z},\\
\rho\frac{\partial^2 w}{\partial t^2} &= \rho Z + \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial w_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial w_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial w_z}\right) + \frac{\partial \tau}{\partial w_x}\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial w_y}\frac{\partial p}{\partial y} + \frac{\partial \tau}{\partial w_z}\frac{\partial p}{\partial z}.
\end{aligned}\right\} \tag{19·5}$$

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These three equations could equally well have been obtained by substituting for $t_{xx}, t_{yy}, \ldots$, in equations (13·2), the expressions given in (8·3) and making use of the operational relationships (17·2).

The undetermined quantity p can be eliminated between the three equations (19·5) and the two equations so obtained, together with the incompressibility condition $\tau = 1$, may be taken as the equations of motion.

Thus, writing equations (19·5) in the form

$$\frac{\partial \tau}{\partial u_x}\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial u_y}\frac{\partial p}{\partial y}+\frac{\partial \tau}{\partial u_z}\frac{\partial p}{\partial z}=\alpha,$$

$$\frac{\partial \tau}{\partial v_x}\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial v_y}\frac{\partial p}{\partial y}+\frac{\partial \tau}{\partial u_z}\frac{\partial p}{\partial z}=\beta$$

and

$$\frac{\partial \tau}{\partial w_x}\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial w_y}\frac{\partial p}{\partial y}+\frac{\partial \tau}{\partial w_z}\frac{\partial p}{\partial z}=\gamma,$$

and solving for $\partial p/\partial x$, $\partial p/\partial y$ and $\partial p/\partial z$, we obtain

$$\frac{\partial p}{\partial x}=\begin{vmatrix} \alpha & \beta & \gamma \\ \dfrac{\partial \tau}{\partial u_y} & \dfrac{\partial \tau}{\partial v_y} & \dfrac{\partial \tau}{\partial w_y} \\ \dfrac{\partial \tau}{\partial u_z} & \dfrac{\partial \tau}{\partial v_z} & \dfrac{\partial \tau}{\partial w_z} \end{vmatrix} \Bigg/ \begin{vmatrix} \dfrac{\partial \tau}{\partial u_x} & \dfrac{\partial \tau}{\partial v_x} & \dfrac{\partial \tau}{\partial w_x} \\ \dfrac{\partial \tau}{\partial u_y} & \dfrac{\partial \tau}{\partial v_y} & \dfrac{\partial \tau}{\partial w_y} \\ \dfrac{\partial \tau}{\partial u_z} & \dfrac{\partial \tau}{\partial v_z} & \dfrac{\partial \tau}{\partial w_z} \end{vmatrix} \tag{19·6}$$

and two similar expressions for $\partial p/\partial y$ and $\partial p/\partial z$.

Now, it can readily be shown by algebraic manipulation that the denominator in the expression on the right-hand side of equation (19·6) is equal to τ^2 and therefore, since $\tau = 1$, to unity. Also, it can be shown that

$$\frac{\partial \tau}{\partial v_y}\frac{\partial \tau}{\partial w_z}-\frac{\partial \tau}{\partial w_y}\frac{\partial \tau}{\partial v_z}=(1+u_x)\,\tau=1+u_x,$$

$$\frac{\partial \tau}{\partial w_y}\frac{\partial \tau}{\partial u_z}-\frac{\partial \tau}{\partial u_y}\frac{\partial \tau}{\partial w_z}=v_x\tau \qquad =v_x$$

and

$$\frac{\partial \tau}{\partial u_y}\frac{\partial \tau}{\partial v_z}-\frac{\partial \tau}{\partial v_y}\frac{\partial \tau}{\partial u_z}=w_x\tau \qquad =w_x,$$

so that equation (19·6) becomes

$$\frac{\partial p}{\partial x}=(1+u_x)\,\alpha+v_x\beta+w_x\gamma. \tag{19·7}$$

In a similar manner it can be shown that

$$\frac{\partial p}{\partial y}=u_y\alpha+(1+v_y)\,\beta+w_y\gamma \tag{19·8}$$

and

$$\frac{\partial p}{\partial z}=u_z\alpha+v_z\beta+(1+w_z)\,\gamma, \tag{19·9}$$

where
$$\alpha = \rho \frac{\partial^2 u}{\partial t^2} - \rho X - \left[\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial u_z}\right)\right],$$
$$\beta = \rho \frac{\partial^2 v}{\partial t^2} - \rho Y - \left[\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial v_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial v_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial v_z}\right)\right],$$
$$\gamma = \rho \frac{\partial^2 w}{\partial t^2} - \rho Z - \left[\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial w_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial w_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial w_z}\right)\right].$$

p can be eliminated from equations (19·7), (19·8) and (19·9) giving
$$\left.\begin{aligned} &\frac{\partial}{\partial y}[(1+u_x)\alpha + v_x\beta + w_x\gamma] = \frac{\partial}{\partial x}[u_y\alpha + (1+v_y)\beta + w_y\gamma] \\ \text{and}\quad &\frac{\partial}{\partial y}[u_z\alpha + v_z\beta + (1+w_z)\gamma] = \frac{\partial}{\partial z}[u_y\alpha + (1+v_y)\beta + w_y\gamma]. \end{aligned}\right\} \tag{19·10}$$

These two equations, together with the relation $\tau = 1$, are the equations of motion for an incompressible material.

Again, the boundary conditions for an incompressible material may be obtained by substituting $W - \lambda f(\tau)$ for W in equations (16·1). Then, writing $-p = \lambda f'(\tau)$, we have
$$\left.\begin{aligned} X_\nu &= \left[\frac{\partial W}{\partial u_x}\cos(x,\nu) + \frac{\partial W}{\partial u_y}\cos(y,\nu) + \frac{\partial W}{\partial u_z}\cos(z,\nu)\right] \\ &\qquad + p\left[\frac{\partial \tau}{\partial u_x}\cos(x,\nu) + \frac{\partial \tau}{\partial u_y}\cos(y,\nu) + \frac{\partial \tau}{\partial u_z}\cos(z,\nu)\right], \\ Y_\nu &= \left[\frac{\partial W}{\partial v_x}\cos(x,\nu) + \frac{\partial W}{\partial v_y}\cos(y,\nu) + \frac{\partial W}{\partial v_z}\cos(z,\nu)\right] \\ &\qquad + p\left[\frac{\partial \tau}{\partial v_x}\cos(x,\nu) + \frac{\partial \tau}{\partial v_y}\cos(y,\nu) + \frac{\partial \tau}{\partial v_z}\cos(z,\nu)\right], \\ Z_\nu &= \left[\frac{\partial W}{\partial w_x}\cos(x,\nu) + \frac{\partial W}{\partial w_y}\cos(y,\nu) + \frac{\partial W}{\partial w_z}\cos(z,\nu)\right] \\ &\qquad + p\left[\frac{\partial \tau}{\partial w_x}\cos(x,\nu) + \frac{\partial \tau}{\partial w_y}\cos(y,\nu) + \frac{\partial \tau}{\partial w_z}\cos(z,\nu)\right]. \end{aligned}\right\} \tag{19·11}$$

Eliminating p, we have
$$\begin{aligned} &\frac{\dfrac{\partial W}{\partial u_x}\cos(x,\nu) + \dfrac{\partial W}{\partial u_y}\cos(y,\nu) + \dfrac{\partial W}{\partial u_z}\cos(z,\nu) - X_\nu}{\dfrac{\partial \tau}{\partial u_x}\cos(x,\nu) + \dfrac{\partial \tau}{\partial u_y}\cos(y,\nu) + \dfrac{\partial \tau}{\partial u_z}\cos(z,\nu)} \\ &\quad = \frac{\dfrac{\partial W}{\partial v_x}\cos(x,\nu) + \dfrac{\partial W}{\partial v_y}\cos(y,\nu) + \dfrac{\partial W}{\partial v_z}\cos(z,\nu) - Y_\nu}{\dfrac{\partial \tau}{\partial v_x}\cos(x,\nu) + \dfrac{\partial \tau}{\partial v_y}\cos(y,\nu) + \dfrac{\partial \tau}{\partial v_z}\cos(z,\nu)} \\ &\quad = \frac{\dfrac{\partial W}{\partial w_x}\cos(x,\nu) + \dfrac{\partial W}{\partial w_y}\cos(y,\nu) + \dfrac{\partial W}{\partial w_z}\cos(z,\nu) - Z_\nu}{\dfrac{\partial \tau}{\partial w_x}\cos(x,\nu) + \dfrac{\partial \tau}{\partial w_y}\cos(y,\nu) + \dfrac{\partial \tau}{\partial w_z}\cos(z,\nu)} = -p, \end{aligned} \tag{19·12}$$

which, together with the incompressibility condition $\tau = 1$, form the boundary conditions for an incompressible material.

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20. The equations of motion and boundary conditions for an incompressible, neo-Hookean material

It has been seen that, for a neo-Hookean material, the stored energy per unit volume W is given in terms of the components of displacement u, v, w in a fixed rectangular, Cartesian co-ordinate system by (9·4).

With this law for the stored energy, α, β and γ, in equations (19·7), (19·8) and (19·9), are given by

$$\left.\begin{aligned} \alpha &= \rho\frac{\partial^2 u}{\partial t^2} - \rho X - \tfrac{1}{3}E\nabla^2 u, \\ \beta &= \rho\frac{\partial^2 v}{\partial t^2} - \rho Y - \tfrac{1}{3}E\nabla^2 v \\ \text{and}\qquad \gamma &= \rho\frac{\partial^2 w}{\partial t^2} - \rho Z - \tfrac{1}{3}E\nabla^2 w \end{aligned}\right\} \qquad (20\cdot 1)$$

and $\partial W/\partial u_x$, $\partial W/\partial u_y$, ..., in equations (19·5), (19·11) and (19·12), are given by (9·5).

Thus, the boundary conditions (19·12) become

$$\frac{\tfrac{1}{3}E[(1+u_x)\cos(x,\nu)+u_y\cos(y,\nu)+u_z\cos(z,\nu)]-X_\nu}{\dfrac{\partial\tau}{\partial u_x}\cos(x,\nu)+\dfrac{\partial\tau}{\partial u_y}\cos(y,\nu)+\dfrac{\partial\tau}{\partial u_z}\cos(z,\nu)}$$

$$= \frac{\tfrac{1}{3}E[v_x\cos(x,\nu)+(1+v_y)\cos(y,\nu)+v_z\cos(z,\nu)]-Y_\nu}{\dfrac{\partial\tau}{\partial v_x}\cos(x,\nu)+\dfrac{\partial\tau}{\partial v_y}\cos(y,\nu)+\dfrac{\partial\tau}{\partial v_z}\cos(z,\nu)}$$

$$= \frac{\tfrac{1}{3}E[w_x\cos(x,\nu)+w_y\cos(y,\nu)+(1+w_z)\cos(z,\nu)]-Z_\nu}{\dfrac{\partial\tau}{\partial w_x}\cos(x,\nu)+\dfrac{\partial\tau}{\partial w_y}\cos(y,\nu)+\dfrac{\partial\tau}{\partial w_z}\cos(z,\nu)} = -p$$

and $$\tau = 1. \qquad (20\cdot 2)$$

By introducing the relations (9·5) in equations (19·5), the equations of motion for an incompressible, neo-Hookean material are obtained as

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= \rho X + \tfrac{1}{3}E\nabla^2 u + \frac{\partial\tau}{\partial u_x}\frac{\partial p}{\partial x} + \frac{\partial\tau}{\partial u_y}\frac{\partial p}{\partial y} + \frac{\partial\tau}{\partial u_z}\frac{\partial p}{\partial z}, \\ \rho\frac{\partial^2 v}{\partial t^2} &= \rho Y + \tfrac{1}{3}E\nabla^2 v + \frac{\partial\tau}{\partial v_x}\frac{\partial p}{\partial x} + \frac{\partial\tau}{\partial v_y}\frac{\partial p}{\partial y} + \frac{\partial\tau}{\partial v_z}\frac{\partial p}{\partial z}, \\ \rho\frac{\partial^2 w}{\partial t^2} &= \rho Z + \tfrac{1}{3}E\nabla^2 w + \frac{\partial\tau}{\partial w_x}\frac{\partial p}{\partial x} + \frac{\partial\tau}{\partial w_y}\frac{\partial p}{\partial y} + \frac{\partial\tau}{\partial w_z}\frac{\partial p}{\partial z}. \end{aligned}\right\} \qquad (20\cdot 3)$$

In addition, the incompressibility condition $\tau = 1$ must be satisfied throughout the material.

The equations (20·3) may be obtained in an alternative form as equations (19·7), (19·8) and (19·9), where α, β and γ are given by equations (20·1). If the body is in equilibrium under the action of no body forces,

$$\alpha = -\tfrac{1}{3}E\nabla^2 u, \quad \beta = -\tfrac{1}{3}E\nabla^2 v \quad \text{and} \quad \gamma = -\tfrac{1}{3}E\nabla^2 w,$$

so that equations (19·7), (19·8) and (19·9) become

$$\left.\begin{aligned} -\partial p/\partial x &= \tfrac{1}{3}E[(1+u_x)\,\nabla^2 u + v_x\,\nabla^2 v + w_x\,\nabla^2 w],\\ -\partial p/\partial y &= \tfrac{1}{3}E[u_y\,\nabla^2 u + (1+v_y)\,\nabla^2 v + w_y\,\nabla^2 w]\\ \text{and}\quad -\partial p/\partial z &= \tfrac{1}{3}E[u_z\,\nabla^2 u + v_z\,\nabla^2 v + (1+w_z)\,\nabla^2 w].\end{aligned}\right\} \quad (20\cdot 4)$$

The equations of motion (20·3) or (20·4) and the boundary conditions (19·12) can readily be transformed into other orthogonal, co-ordinate systems for the solution of problems, for which a rectangular, Cartesian system is not suitable.

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